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TASK ANALYSIS IN CURRICULUM DESIGN: A HIERARCHICALLY SEQUENCED INTRODUCTORY MATHEMATICS CURRICULUM[1]

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A method of systematic task analysis is applied to the problem of designing a sequence of learning objectives that will provide an optimal match for the child's natural sequence of acquisition of mathematical skills and concepts. The authors begin by proposing an operational definition of the number concept in the form of a set of behaviors which, taken together, permit the inference that the child has an abstract concept of "number". These are the "objectives" of the curriculum. Each behavior in the defining set is then subjected to an analysis that identifies hypothesized components of skilled performance and prerequisites for learning these components. On the basis of these analyses, specific sequences of learning objectives are proposed. The proposed sequences are hypothesized to be those that will best facilitate learning, by maximizing transfer from earlier to later objectives. Relevant literature on early learning and cognitive development is considered in conjunction with the analyses and the resulting sequences. The paper concludes with a discussion of the ways in which the curriculum can be implemented and studied in schools. Examples of data on individual children are presented, and the use of such data for improving the curriculum itself, as well as for examining the effects of other treatment variables, is considered.

The curriculum presented in this paper is an intermediate result of a research program exploring application of detailed task-analysis procedures to the problem of designing sequences of learning objectives. The aim of this research program is to develop a systematic method of specifying and validating learning hierarchies so that instructional programs can be designed that provide an optimal match for a child's natural sequence of acquisition. It is assumed that curricula that closely parallel this sequence will facilitate learning under a wide variety of specific teaching methods.

The basic rationale for the methods employed here has been presented in papers by Resnick (1967) and by Resnick and Wang (1969). Briefly, the strategy is to develop hierarchies of learning objectives such that mastery of objectives lower in the hierarchy (simpler tasks) facilitates learning of higher objectives (more complex tasks), and ability to perform higher-level tasks reliably predicts ability to perform lower-level tasks. This involves a process of task analysis in which specific behavioral components are identified and prerequisites for each of these determined (*cf.* Gagne, 1962, 1968). Detailed procedures of analysis are explicated in the course of this paper.

An introductory mathematics curriculum must present the fundamental concepts of mathematics, or operations leading to them, in forms simple enough to be learned by very young children. Methodologically, this requires that target concepts be identified, and that hierarchies of specific objectives then be constructed to guide the child from naivete to competence in understanding and using these concepts. Finally, empirical studies, both laboratory and classroom, must be undertaken to validate the sequences of objectives and study the functioning of the curriculum in an applied setting. The first two sections of this paper deal with the problems of defining and analyzing early math-

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ematical content. The final section describes a program of classroom research in which the characteristics of the behaviorally derived curriculum are examined.

CONTENT OF AN INTRODUCTORY MATHEMATICS CURRICULUM

The Concept of Number

One of the main goals of the mathematics curriculum reform movement during the past decade has been to present mathematics as a body of knowledge that obeys well-defined principles or laws. Emphasis on the inherent structure of mathematics can be seen throughout the curricula and writings of various groups of reformers (*e.g.,* Cambridge Conference on School Mathematics, 1963; DeVault and Kriewall, 1969). At the heart of the structures present in school mathematics are the concepts of sets, relations, and numbers. In the early years of a child's mathematical education, the curricula emphasize experiences designed to foster the concept of number. With the acquisition of the number concept, the child is prepared to advance to the operations on natural numbers, and to study the properties of these operations. The structure of the natural numbers, then, is one of the central concerns of mathematics curricula throughout elementary school.

To a mathematician, the concept of natural number is the common property shared by all sets that are in a one-to-one correspondence with each other. Thus, the concept of the natural (or cardinal) number "two" is derived from the (only) property that is shared by all sets in a one-to-one correspondence with, for instance, the set {a, b}. This property is called the number "two"; as a generalization, it is the concept "two". Other natural numbers are defined in a similar manner.

While the concept of number is clearly defined mathematically, it is not at all clear how a child attains the concept, or even what kinds of performance signify such attainment. Traditional arithmetic has stressed the learning of such skills as counting objects, using written numerals, and, later, calculating. Both Piaget-oriented researchers in mathematics learning (*e.g.,* Dienes, 1966, 1967; Lovell, 1966) and developmental psychologists (*e.g.,* Flavell, 1963; Kohlberg, 1968; Wohlwill, 1960) focus instead on processes that reflect more directly the mathematical definition of the number concept. Mathematicians stress logical relations among ordered sets, and particularly the notion of one-to-one correspondence among sets. New math curricula are intended to provide the child with the experiences with sets and logic that will directly develop these concepts. Piaget adds to the mathematicians' concern a special emphasis on seriation, on the child's recognition of invariance of number across spatial transformations (conservation), and on the correspondence of ordinal and cardinal number (Piaget, 1965).

The basic goal of the present mathematics curriculum is the development in children of a stable concept of number. Many developmental psychologists are skeptical of the possibility of directly teaching these concepts, stressing instead the role of "general experience" in inducing the state of "concrete operations", which includes mathematical operations along with classificatory logic and related concepts (Kohlberg, 1968). Our work, however, operates from a broad assumption that operational number concepts can be taught, believing that "general experience" is in fact composed of a multiplicity of specific experiences, certain ones of which are critical in the acquisition of an operational number concept. The problem, both for psychological research and educational design, is to discover which experiences are the crucial ones; that is, which early behaviors form the building blocks of the higher-level competence one seeks to establish.

Behavioral Definition of the Number Concept

The first step in developing a hierarchy of curriculum objectives leading to an operational concept of number was to specify in behavioral terms a number of specific components of the number concept. The behaviors thus specified comprise an operational definition of the number concept in the form of concrete performances, which, taken together, permit the inference that the child has an abstract concept of number. Some of the behaviors relate directly to the mathematical-psychological definition of number; others are associated with common symbols for numbers. These behaviors comprise the actual objectives of the curriculum.

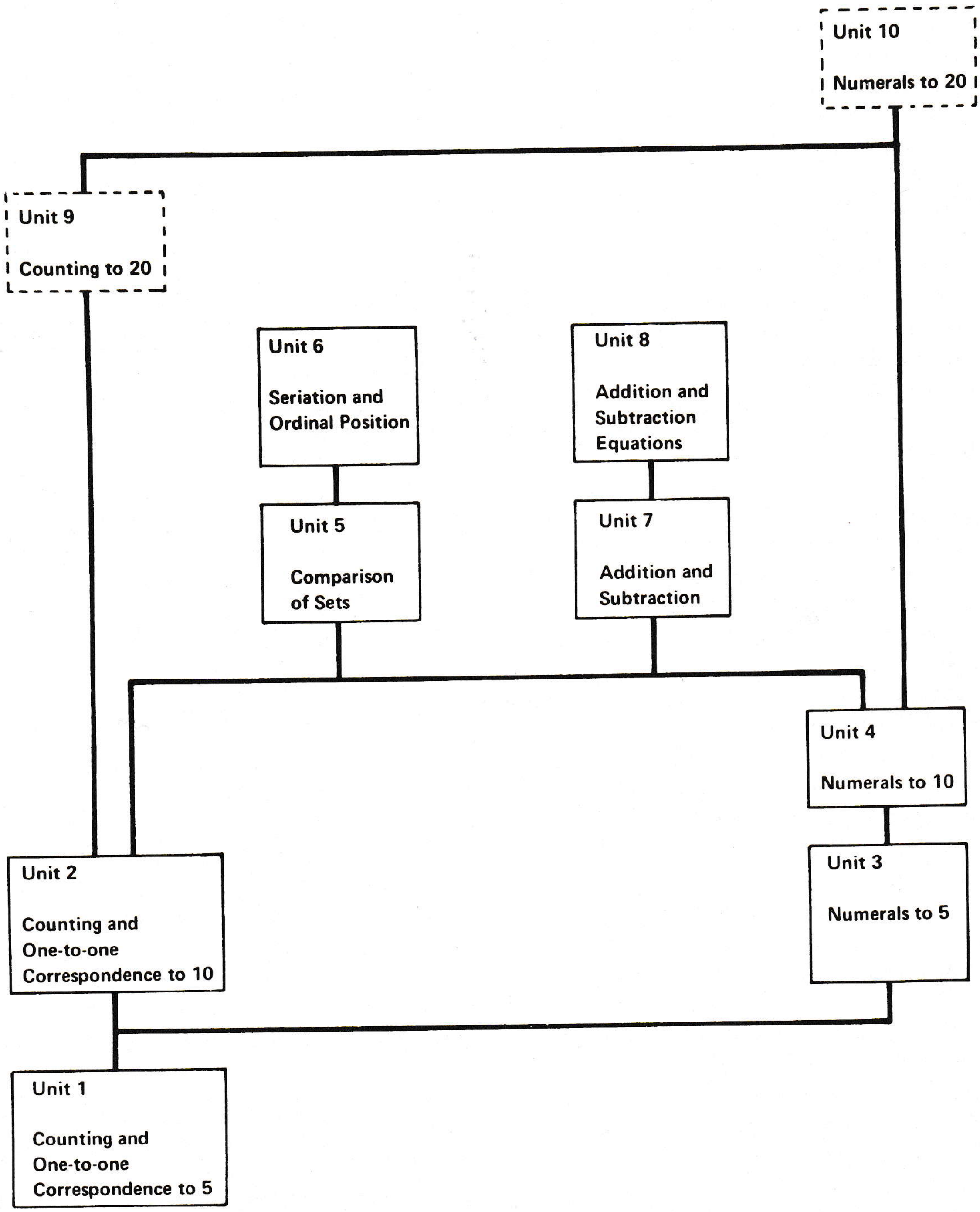

Fig. 1. Hierarchical sequence of introductory mathematics units.

 But...

There are eight units in the introductory curriculum, each made up of a series of specific objectives. Units 1 and 2 cover counting skills to 10, and simple comparison of sets by one-to-one correspondence. Units 3 and 4 cover the use of numerals. Units 5 and 6 include more complex processes of comparing and ordering sets. Unit 7 introduces the processes of addition and subtraction, while Unit 8 uses equations to establish more sophisticated understanding of partition and combination of sets. The numbering of the units is for reference purposes, and does not imply a linear order of instruction. Figure 1 shows the pattern of hierarchical relationships among the units and the order in which they can be presented without skipping prerequisites. Two higher-level units (9 and 10), which are not discussed or analyzed in the present paper, are also shown in the figure, because knowledge of the position of these units in the hierarchy is necessary for interpretation of the empirical data presented later in the paper.

In determining possible teaching sequences, the charts are read from the bottom up. The earliest units appear at the bottom and are considered prerequisite to those appearing above and connected by a line. Unit 1, for example, is prerequisite to 2 and 3; and 3 is prerequisite to 4. Units 2 and 3, however, have no prerequisite relation to each other, and can be taught in either order. Unit 5 has two prerequisites, 2 and 4, and according to this analysis would not normally be taught until both of these units were mastered.

The division of the curriculum into units was based on considerations of educational practice rather than on mathematical theory or analysis. In general, the aim was to establish units that would maximize the child's experience of success and also make for relative ease of administration in an individualized classroom. These criteria explain, for example, the decision to break the initial introduction of counting skills into two units, one for sets up to five (Unit 1), and the second for sets up to 10 (Unit 2). The use of written numerals (Units 3 and 4) is treated as a separate group of objectives, largely because of classroom and experimental evidence that counting is learned earlier than written-numeral presentation and that learning the numerals is easier once counting is well established (Wang, Resnick, and Boozer, 1971).

Table 1 lists the objectives that comprise the current curriculum. Each objective listed defines a terminal objective of the curriculum—an objective deemed important enough to be subjected to direct measurement in assessment of the child's progress through the curriculum. Figure 2 shows the hierarchical relationship between the specific objectives in each unit. The completed curriculum in use in our classrooms includes a heavy emphasis on classification skills and concepts (including multiple relations, sorting, intersection of sets, *etc.*). Such skills and concepts are recognized as likely prerequisites for full mathematical understanding, but have not been included directly in the mathematics curriculum. Instead, they appear in separate "classification and language" sequences that can be implemented before or simultaneously with the mathematics curriculum. *

Williams

ANALYSIS AND SEQUENCING OF THE OBJECTIVES

The ordering of objectives within each unit is based on detailed analyses of each task. These analyses are designed to reveal component and prerequisite behaviors for each terminal objective, both as a basis for sequencing the objectives and to provide suggestions for teaching a given objective to children who are experiencing difficulty. The detailed analyses identify many behaviors that are not part of the formal curriculum, but which underlie the stated objectives and may need to be taught explicitly to some children. Often, two superficially similar tasks differ with respect to their demands on some basic function such as memory or perceptual organization. These differences between tasks provide the basis for ordering tasks according

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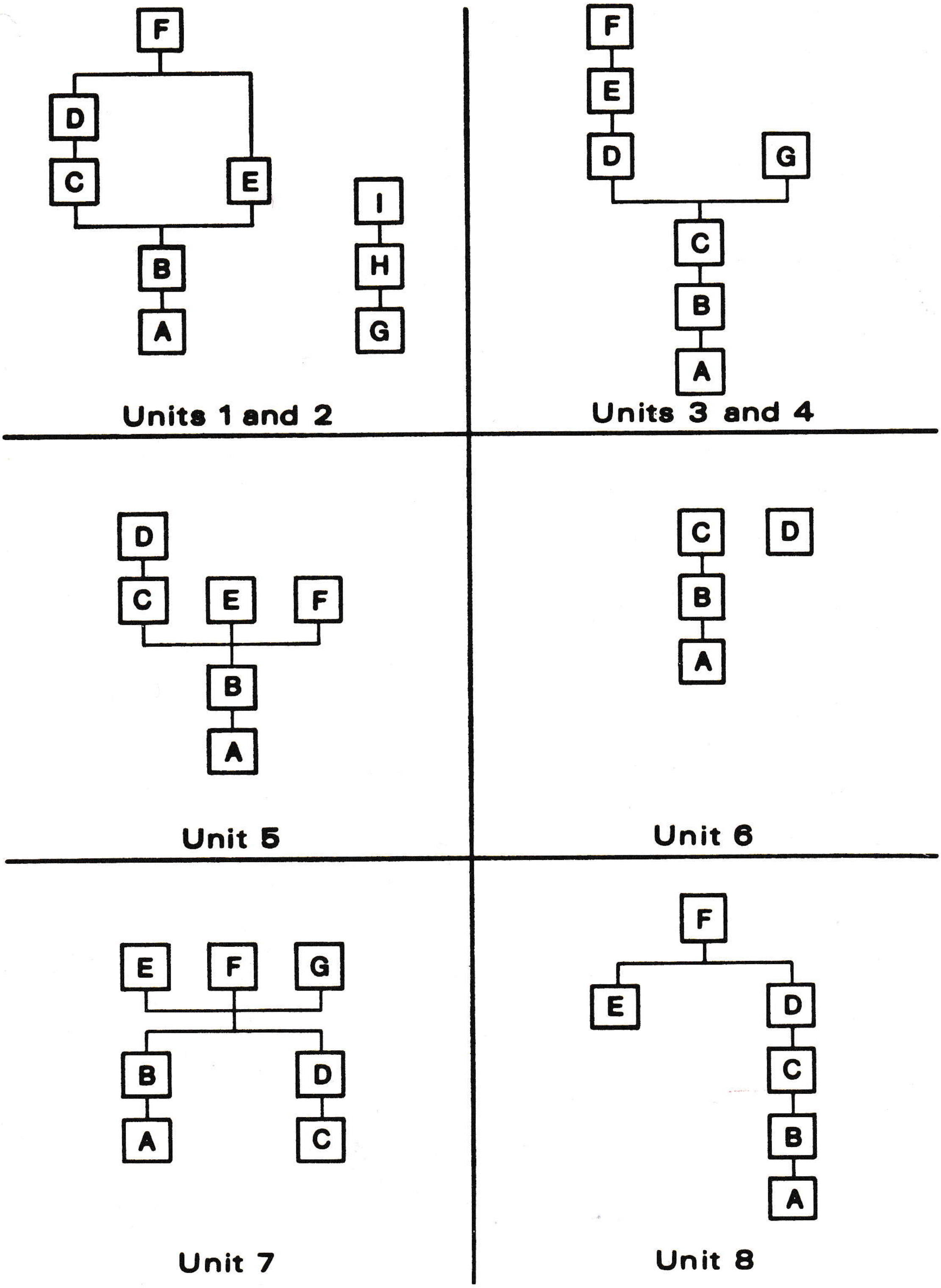

Fig. 2. Hierarchical sequence of individual objectives, by unit.

Table 1

Objectives of the Curriculum

Units 1 and 2: Counting and One-to-One Correspondence[a]

A. The child can recite the numerals in order.
B. Given a set of moveable objects, the child can count the objects, moving them out of the set as he counts.
C. Given a fixed ordered set of objects, the child can count the objects.
D. Given a fixed unordered set of objects, the child can count the objects.
E. Given a numeral stated and a set of objects, the child can count out a subset of stated size.
F. Given a numeral stated and several sets of fixed objects, the child can select a set of size indicated by numeral.
G. Given two sets of objects, the child can pair objects and state whether the sets are equivalent.
H. Given two unequal sets of objects, the child can pair objects and state which set has more.
I. Given two unequal sets of objects, the child can pair objects and state which set has less.

cf. Williams

Units 3 and 4: Numerals[b]

A. Given two sets of numerals, the child can match the numerals.
B. Given a numeral stated and a set of printed numerals, the child can select the stated numeral.
C. Given a numeral (written), the child can read the numeral.
D. Given several sets of objects and several numerals, the child can match numerals with appropriate sets.
E. Given two numerals (written), the child can state which shows more (less).
F. Given a set of numerals, the child can place them in order.
G. Given numerals stated, the child can write the numeral.

Unit 5: Comparison of Sets

A. Given two sets of objects, the child can count sets and state which has more objects or that sets have same number.
B. Given two sets of objects, the child can count sets and state which has less objects.
C. Given a set of objects and a numeral, the child can state which shows more (less).
D. Given a numeral and several sets of objects, the child can select sets which are more (less) than the numeral; given a set of objects and several numerals, the child can select numerals which show more (less) than the set of objects.
E. Given two rows of objects (not paired), the child can state which row has more regardless of arrangement.
F. Given three sets of objects, the child can count sets and state which has most (least).

Unit 6: Seriation and Ordinal Position

A. Given three objects of different sizes, the child can select the largest (smallest).
B. Given objects of graduated sizes, the child can seriate according to size.
C. Given several sets of objects, the child can seriate the sets according to size.
D. Given ordered set of objects, the child can name the ordinal position of the objects.

Unit 7: Addition and Subtraction (sums to 10)

A. Given two numbers stated, set of objects, and directions to add, the child can add the numbers by counting out two subsets then combining and stating combined number as sum.
B. Given two numbers stated, set of objects, and directions to subtract, the child can count out smaller subset from larger and state remainder.
C. Given two numbers stated, number line, and directions to add, the child can use the number line to determine sum.

[a]Unit 1 involves sets of up to five objects; unit 2 involves sets of up to 10 objects.
[b]Unit 3 involves numerals and sets of up to five objects; unit 4 involves numerals and sets of up to 10 objects.

D. Given two numbers stated, number line, and directions to subtract, the child can use number line to subtract.
E. Given addition and subtraction word problems, the child can solve the problems.
F. Given written addition and subtraction problems in form: $\begin{array}{r} x \\ \underline{+y} \end{array}$ or $\begin{array}{r} x \\ \underline{-y} \end{array}$; the child can complete the problems.
G. Given addition and subtraction problems in form: $x + y = \square$, or $x - y = \square$; the child can complete the equations.

Unit 8: Addition and Subtraction Equations

A. Given equation of form $z = \square + \triangle$, the child can show several ways of completing the equation.
B. Given equation of form $x + y = \quad + \quad$, the child can complete the equation in several ways.
C. Given equations of forms $x + y = z + \quad$ and $x + y = \quad + z$, the child can complete the equations.
D. Given equations of forms $x + \square = y$ and $\square + x = y$, the child can complete the equations.
E. Given complete addition equation (*e.g.,* $x + y = z$), the child can write equations using numerals and minus sign (*e.g.,* $z - x = y$) and demonstrate relationship.
F. Given counting blocks and/or number line, the child can make up completed equations of various forms.

to complexity, and thus for predicting optimal instructional sequences. The detailed rationale for such sequencing will be described in the following sections, which discuss each of the units in some detail. Figures showing the detailed analyses of some of the objectives are included in order to exemplify the method of analysis. The full set of analyses are available from the authors.[2]

To interpret the figures that follow, it is necessary to understand the procedures followed in performing the analyses and the conventions used in displaying them. In each of Figures 3 to 14, the top box contains a statement of the objective being analyzed. In this box, and throughout the analysis charts, the entry above the line describes the stimulus situation with which the child will be presented, and the entry below the line describes the child's response. Thus, in Figure 3, box Ia should be read as, "*Given* a set of movable objects, *the child can* count objects, moving them out of set as he counts". Box IIIa would be read, "*Given* a set of objects, *the child can* synchronize touching an object and saying a word". Adherence to this convention assures that each box in the analysis will contain a behaviorally defined task, one that can be tested by direct observation.

The first step in performing an analysis is to describe in as much detail as possible the actual steps involved in skilled performance of the task. The analyses generated share certain features of "process models" used in studies of computer simulation of thinking (see Newell and Simon, 1972; Klahr and Wallace, 1970), but are less formalized. The results of this "component analysis" are shown in level II of each chart. The double lines around the boxes indicate that these behaviors are *components* of the terminal behavior; it is hypothesized that the skilled person *actually performs* these steps (although sometimes very quickly and covertly) as he performs the terminal task. The arrows between the boxes indicate that the component behaviors are performed in a temporal sequence. Sometimes (*e.g.,* Figure 3) there are "loops" in the chain, indicating that it is necessary to recycle through some of the steps several

[2] Write to Lauren B. Resnick requesting a copy of "Behavior Analysis and Curriculum Design: A Hierarchically Sequenced Introductory Mathematics Curriculum," Learning Research and Development Center, University of Pittsburgh, Pittsburgh, Pa. A charge of $1.00 covers the cost of printing and handling.

times to complete the task. Where a box is divided vertically, a choice or decision point in the task is indicated. For example, as structured in Figure 7, box IId shows a point at which either of two different responses might well be appropriate, depending on whether two numbers are found to be the same or different.

Components vs. Prerequisites

Once the components are identified, a second stage of analysis begins. Each component that has been specified is now considered separately, and the following question asked: "In order to perform this behavior, which simpler behavior(s) must a person be able to perform?" Here, the aim is to specify *prerequisites* for each of the behaviors. Prerequisite behaviors, in contrast to component behaviors, are not actually performed in the course of the terminal performance. However, they are thought to *facilitate* learning of the higher level skill. More precisely, if A is prerequisite to B, then learning A first should result in positive transfer when B is learned, and anyone able to perform B should be able to perform A as well. The first set of prerequisites appear in level III of each of the charts.

Continuing the analysis, identified prerequisites are themselves further analyzed to determine still simpler prerequisite behaviors. This can result in charts showing several levels of prerequisites, with complex interrelationships among the behaviors (*e.g.*, Figure 11). Analysis stops when a level of behavior is reached that can be assumed in most of the student population in question, or when another terminal behavior in the set under analysis appears as a prerequisite. In the latter case, reference is made to the analysis of that behavior (*e.g.*, Figure 5, box IIIa). Sometimes a single behavior is prerequisite to more than one higher-level behavior. Conversely, a given component or prerequisite can have more than a single prerequisite. In reading the charts, it is necessary to remember simply that a given behavior is prerequisite to all behaviors above it and connected to it with a line.

Counting: Units 1 and 2

Units 1 and 2 specify several different kinds of counting behavior (Objectives A to F). Analyses of these behaviors (Figures 3 to 7) suggest that each type of counting task has certain unique components and prerequisites. Because the tasks are behaviorally different, they have been included as separate objectives in the curriculum.

Fixed vs moveable

Figure 3 shows the analysis for Objective 1-2:B, counting a set of moveable objects. The key component is moving an object out of the set while saying a numeral (boxes IIa and IIb). This behavior has two prerequisites: synchronizing touches with counts (box IIIa) and reciting the numerals in order (box IIIb). Because he can move objects out of the set as he counts them, the child has no problem of remembering which objects have been counted. In counting a fixed set (Objective C; Figure 4), on the other hand, the child must touch the objects in a fixed pattern in order not to miss any objects nor touch any of them twice (*cf.* Potter and Levy, 1968). This additional prerequisite is shown in Figure 4 in box IIIc. Since Objective C has all the prerequisites of B plus an additional one, C was placed above B in the unit hierarchy (see Figure 2). This indicates a hypothesis that learning B first will facilitate the learning of C.

Objective D (Figure 5) adds still another new component. When the objects to be counted are physically scattered (unordered), rather than lined up in a row or other recognizable pattern, the task of keeping track of which objects have been touched is considerably more difficult. Beckwith and Restle (1966) presented data suggesting that this problem is typically solved by first visually grouping or patterning the objects, and then counting as if the set had been ordered to begin with. Figure 5 (box IIa) shows this behavior of visual grouping as a component of counting unordered sets. Box IIb on this chart describes a behavior equivalent to counting an ordered set, and the reader is re-

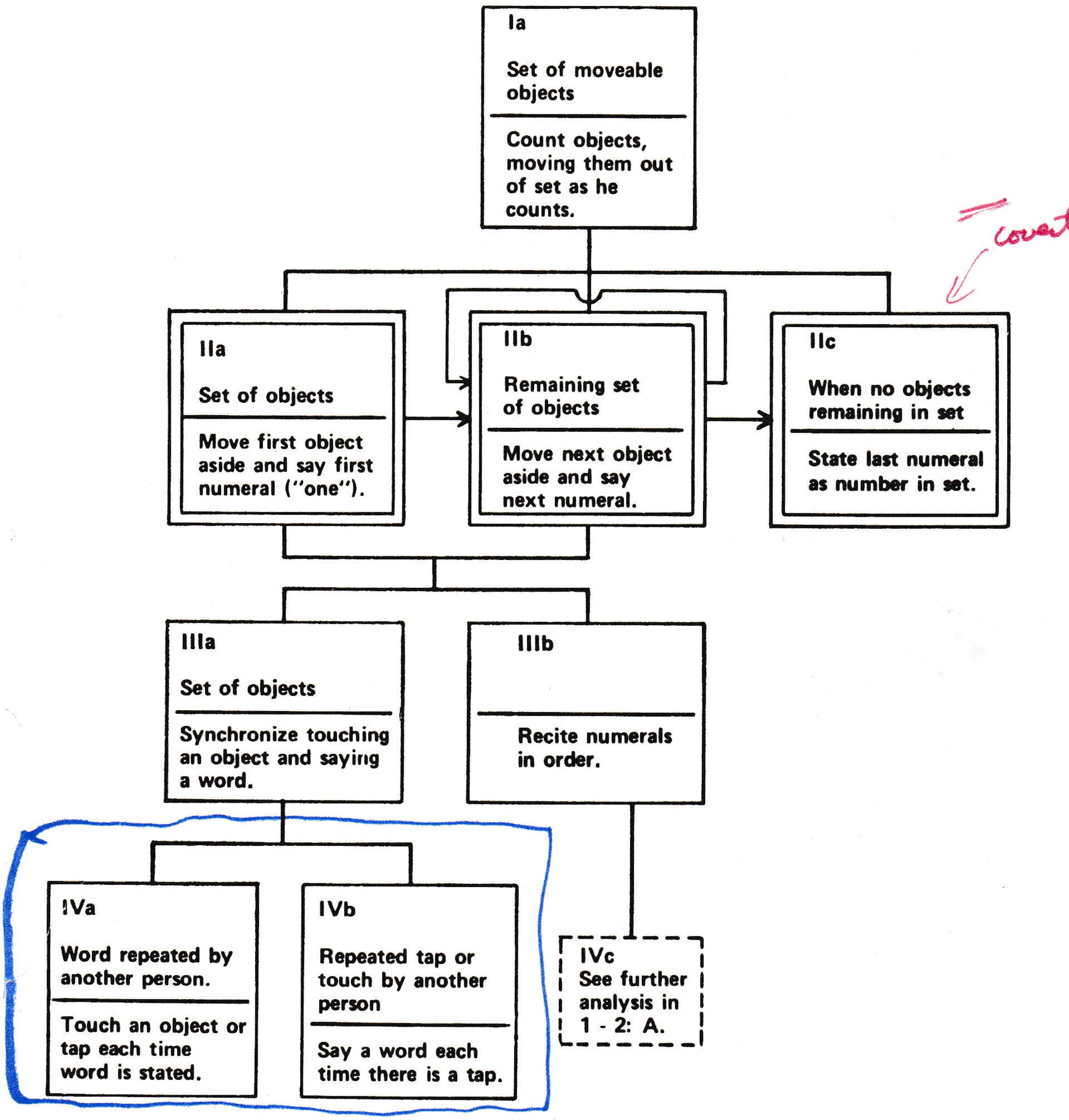

Fig. 3. Analysis of Objective 1-2:B.

ferred to Objective 1-2:C for further analysis. Since C appears as a prerequisite to D in the analysis, Objective D appears above C in Units 1 and 2.

Objective E (Figure 6), counting out a subset from a larger set, returns to the use of moveable objects, as in Objective B. However, whereas in B the child simply continues counting until the set is exhausted, in E he must remember the number of the subset he has been asked for (box IIa) and stop when he reaches that number (IIc). Figure 6, therefore, shows Objective 1-2:B as a prerequisite to E (box IIa), and this dependency is reflected in the unit hierarchies.

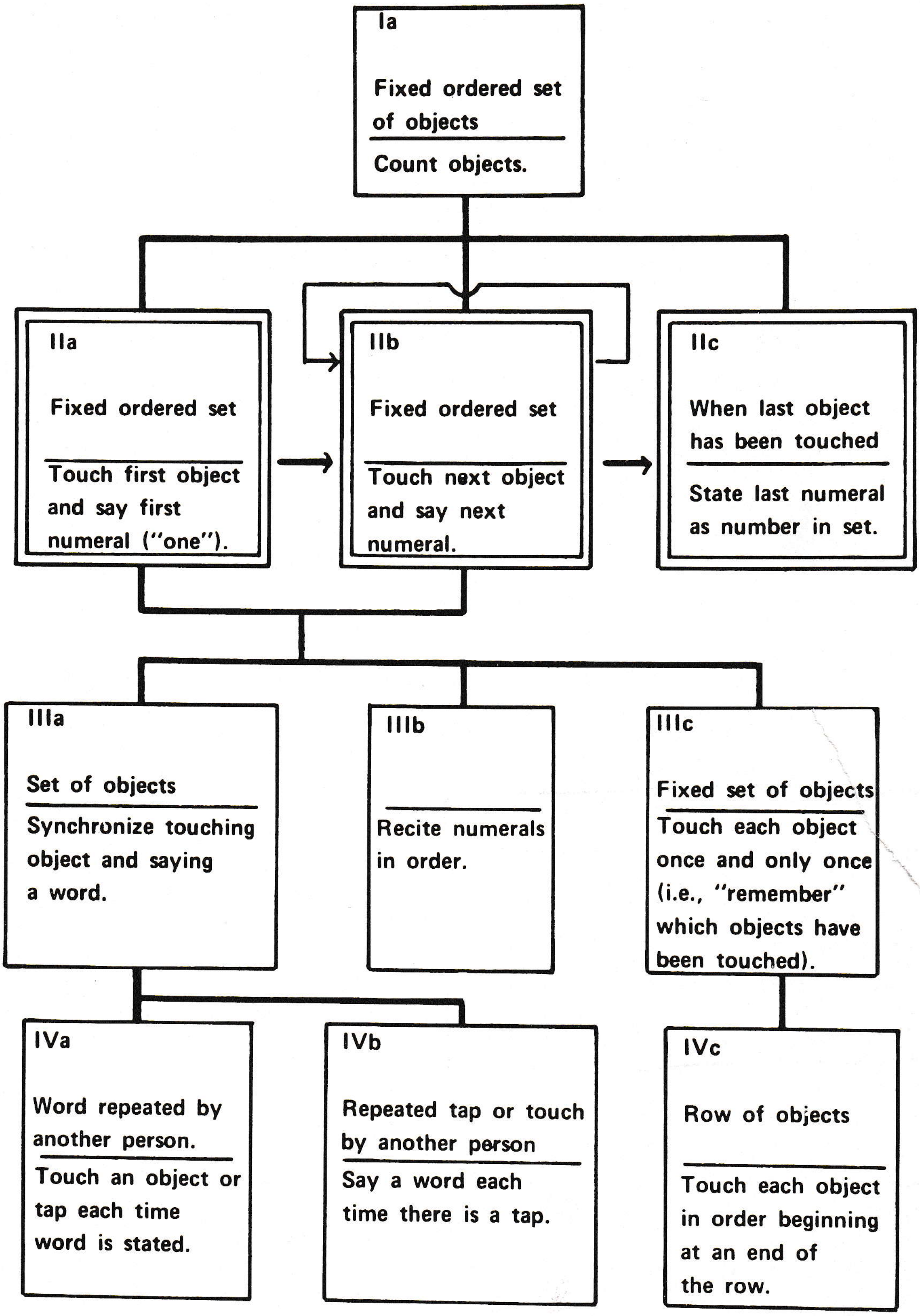

Fig. 4. Analysis of Objective 1-2:C.

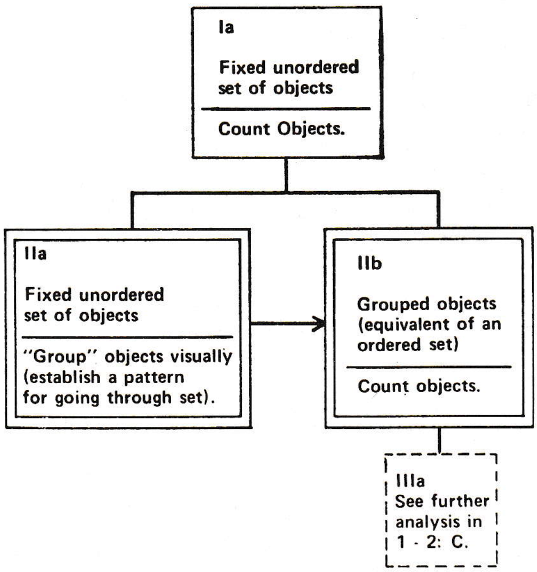

Fig. 5. Analysis of Objective 1-2:D.

Counting out a subset does not share with counting fixed arrays the component of keeping track of which objects have been counted. For this reason, the unit charts show E as independent of C and D. Objective F (Figure 7), on the other hand, has both the memory component (boxes IIa and IIc) similar to that in E, and the component of counting fixed arrays (box IIb), as in C and D. For this reason, the unit hierarchies suggest that Objective F not be introduced until both the C-D sequence and E have been learned.

At the same time as he is learning to count, the child can be working on another basic aspect of the number concept, one-to-one correspondence. In Objectives G, H, and I, he learns to pair objects from two sets to determine whether the sets are equivalent or which set has more (or less) objects. The analyses of Objectives G ("equivalent") and H ("more") showed nearly identical components. The child must: (a) pair the objects, one from each set; (b) decide whether there are extra (*i.e.,* unpaired) objects in either set; and (c) if there are no extra objects, state that the sets are equivalent; *or* if there are extra objects, state that the sets are not equivalent. The only difference among the three objectives appears in the third component. To determine which set has more objects, the child must correctly select the set with extra objects, while, to decide whether the sets are equivalent, he need only determine whether there are extra objects in *either* set. On the basis of this slight additional complexity, Objective H was placed above G in the unit hierarchies.

To determine which of two sets has fewer objects (Objective I), it is necessary to determine which set has extra objects, and then choose the other set. This is behaviorally analogous to using negative information, which is known to be difficult for young children. Thus, the task analysis suggested that the concept "less" should be more difficult to learn than the concept "more". For this reason, Objective I was placed above H in the unit hierarchy, yielding a predicted learning sequence for one-to-one correspondence tasks in which "equivalent" (G) is prerequisite to "more" (H), which is in turn prerequisite to "less" (I).

The sequence G-H-I is supported empirically in a study by Uprichard (*unpublished*) *in* which "equivalent to", "greater than", and "less than" was shown to be the optimal order for teaching these three concepts. On the other hand, data from a scaling study by Wang (1973) suggest that preschool children normally learn the concept "more" before they learn "equivalent". as a mand.

Thus, there is some doubt as to the appropriate sequence for Objectives G and H; it may, in fact, be likely that both objectives will be learned most easily when taught simultaneously, as "contrast" cases for one another. The Uprichard and the Wang findings are in agreement concerning the dependency of the concept of "less than" on "more" and "equivalent". In addition, Donaldson and Balfour (1968) found that children at about age four typically respond to the term "less" as if it were synonymous with "more". Thus, for this concept, existing empirical data support the predictions derived from task analysis.

Will free-operant object-naming facilitate rational counting?

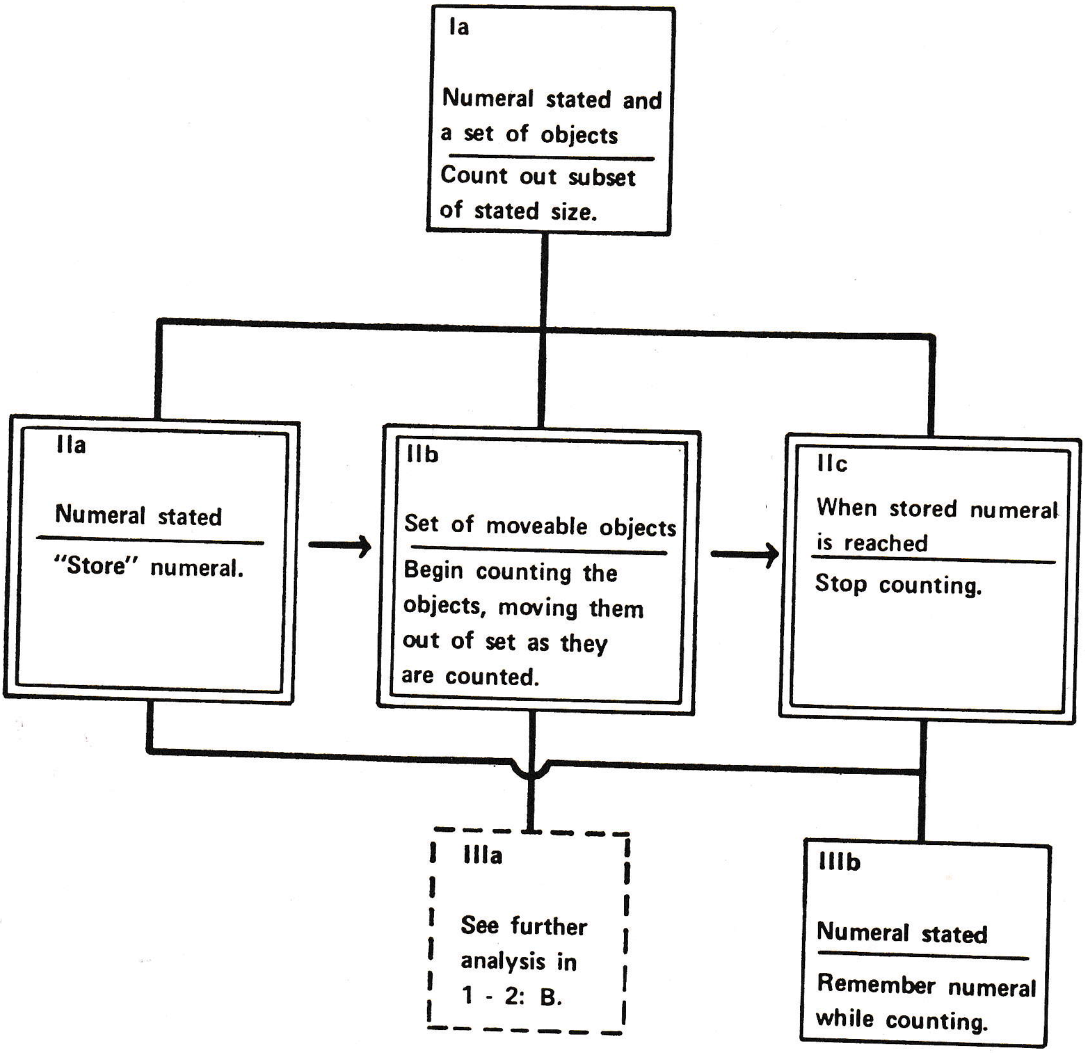

Fig. 6. Analysis of Objective 1-2:E.

Numerals: Units 3 and 4

Units 3 and 4 introduce written numerals. Objectives A, B, and C in each unit establish the basic skills of recognizing and reading numerals. The sequence of matching (A), identifying (B), and naming (C) numerals is a basic sequence for teaching the names of a set of objects. It is used elsewhere in our program for teaching labels such as color, geometric shapes, names of common objects, *etc.* This sequence has been empirically validated in two separate studies (Wang, 1973; Wang, *et al.*, 1971).

Objectives D through F are intended to ensure that the child attaches meaning to the written symbols. In D, he matches sets with numerals, thus combining counting and numeration skills. In E, the child compares numerals for size of the sets represented. The analysis of this objective showed as prerequisites counting out a set of the size indicated by a numeral and comparing sets by one-to-one correspondence. Neither of these behaviors is a component in the sense that skilled persons would actually perform them in the process of comparing numerals. However, they are the processes that logi-

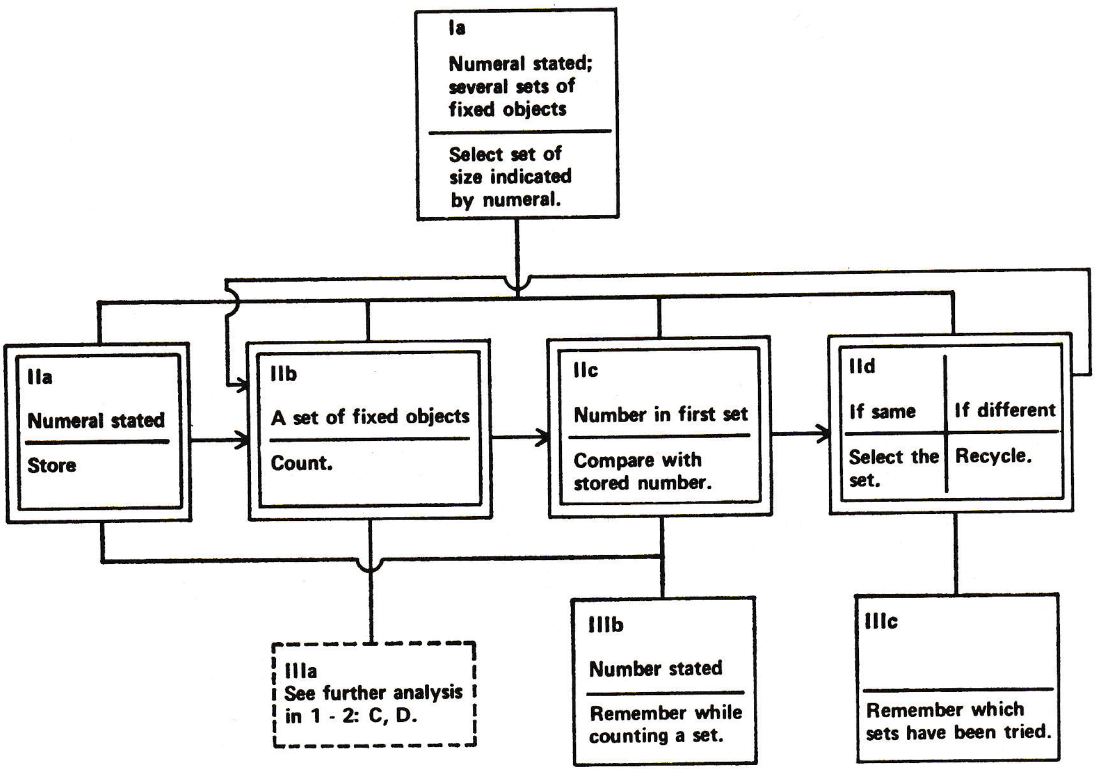

Fig. 7. Analysis of Objective 1-2:F.

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cally underlie the assignment of relative value to numerals, and, therefore, represent prerequisites to performing the terminal task with comprehension, rather than purely algorithmically. They are also prerequisites in the sense that a skilled person undertaking to explain the process to a novice would probably demonstrate these behaviors.

Objective F requires ordering a set of numerals. Two different methods of performing this task have been identified. The first method involves placing the lowest numeral first, then the next lowest, and the next, until the set of numerals is exhausted. The second method is to order two numerals, then arrange a third numeral with respect to the first two, and continue inserting new numerals into the series by a process of successive comparison. An elementary form of transitivity (see Murray and Youniss, 1968; Smedslund, 1963) seems to be involved in this latter method since a numeral is placed as soon as a single higher numeral is found, and comparison with the rest of the numerals higher in the series is not required. This method is also more complicated with respect to maintaining a spatial arrangement and keeping track of which positions have been tested. The problem of creating ordered series is encountered frequently in mathematics. In the present curriculum, it appears again in Unit 6 where seriation by size and numerosity appear. Detailed analyses of methods of seriation are presented in that context.

Comparison of Sets: Unit 5

Units 5 and 6 are the points at which the child begins to combine his skills in counting, one-to-one correspondence, and numeration into an integrated, operational number concept. In Objectives A and B of Unit 5, he learns a new method of comparing set size, this time by counting the sets and comparing the numerals

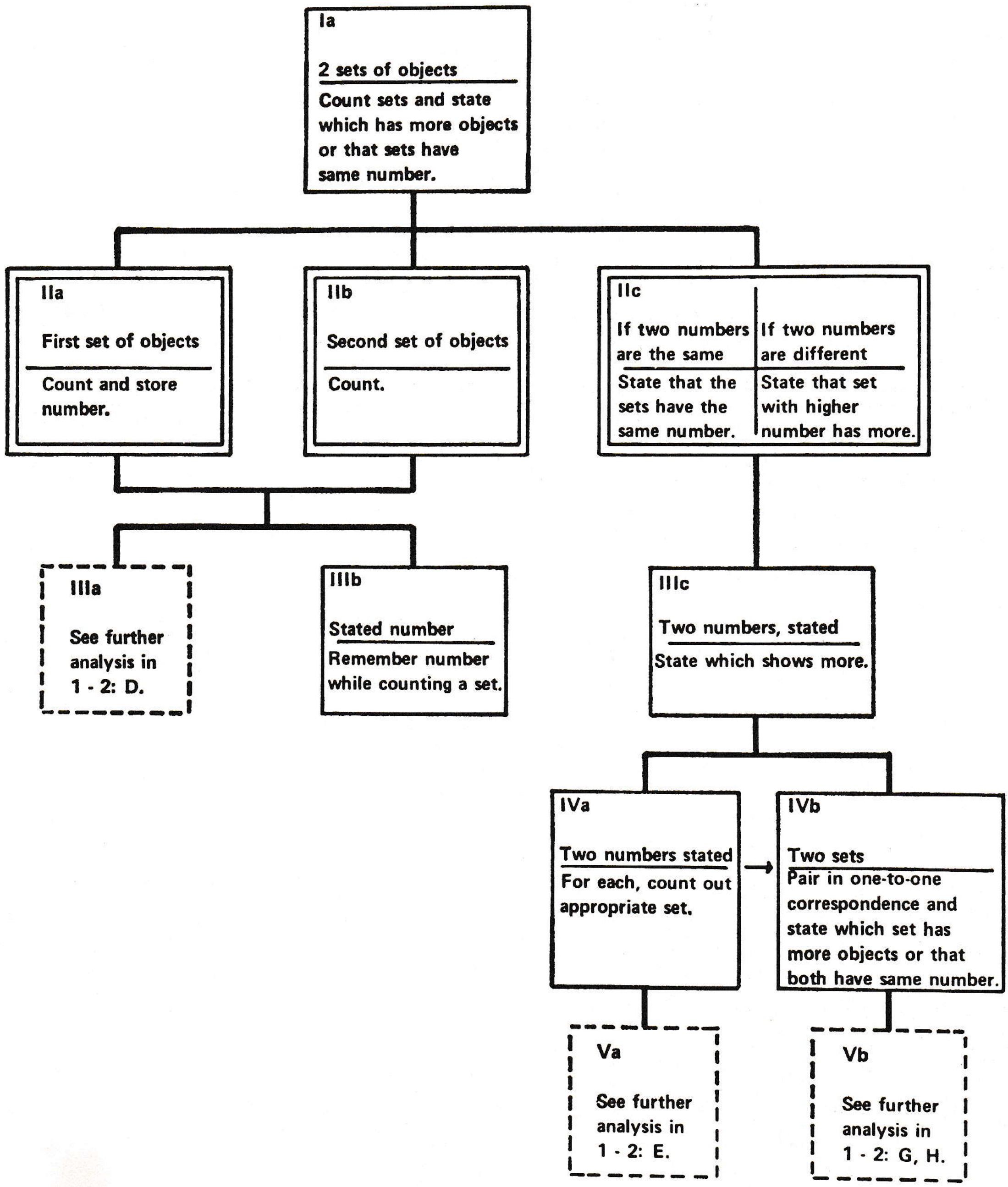

Fig. 8. Analysis of Objective 5:A.

stated. Analyses of these objectives, in Figures 8 and 9 show comparison of sets by one-to-one correspondence as a prerequisite (boxes IVa and IVb in both figures). While it would probably be possible for a child to learn to count and compare without being able to perform one-to-one correspondence operations, his comprehension of the nature of number comparison would be in doubt in such a case. By specifying one-to-one correspondence as a prerequisite, the curriculum ensures that children will relate their counting operations to the basic mathematical definition of number. Thus, specification of the process that *logically* underlies the performance

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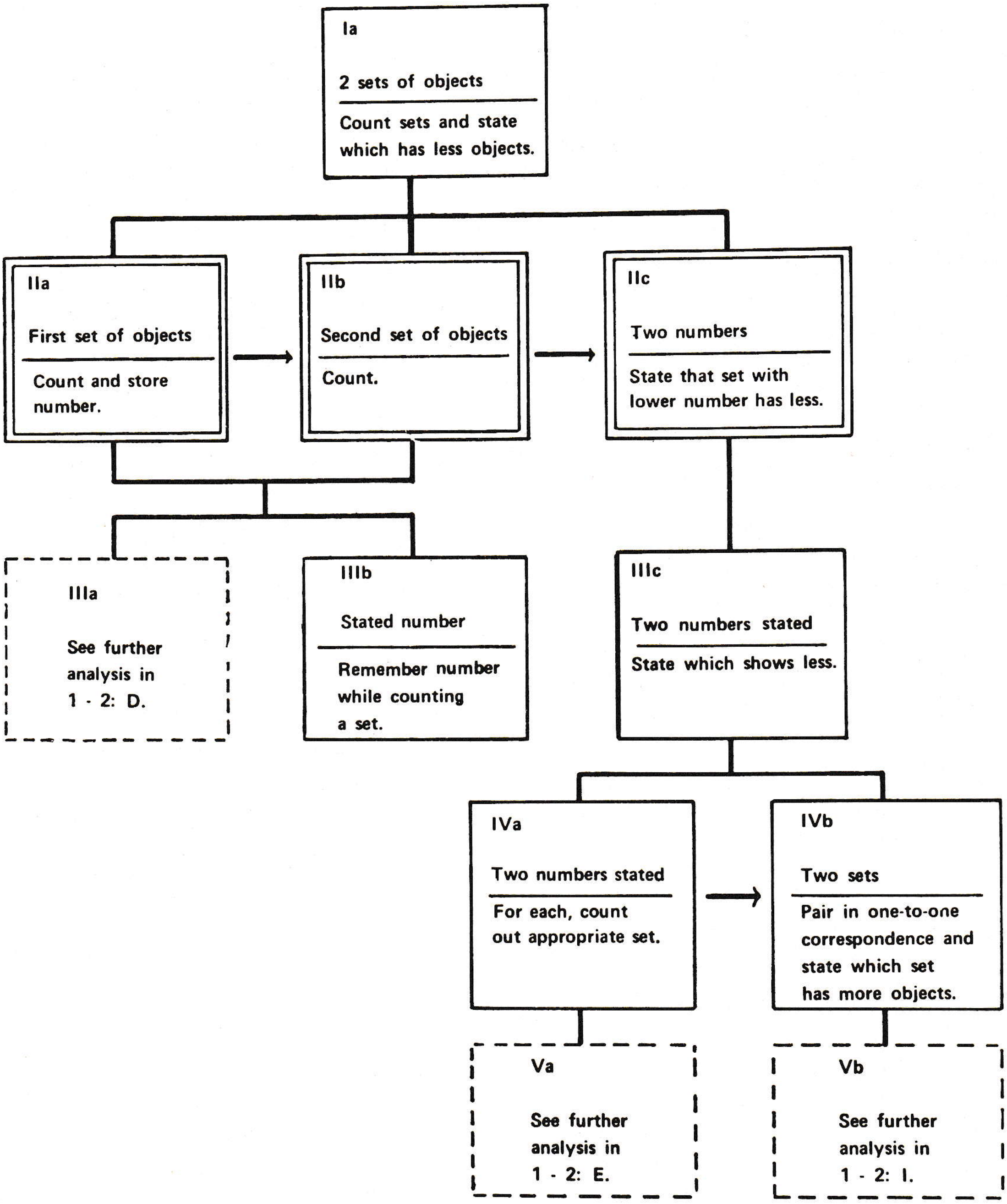

Fig. 9. Analysis of Objective 5:B.

being learned as a prerequisite helps to assure that the new performance will not be learned purely as an algorithm.

Objectives 5:C and 5:D require the comparison of a set with a numeral. This represents a consolidation of numeration skills taught in Units 3 and 4 and their integration with the concepts of set size and set comparison. These objectives have as prerequisites reading numerals (3-4:C), counting sets (1-2:D), comparison of sets (5:A and 5:B), and comparison of numerals (3-4:E). Since comparison of sets and of numerals are combined in a single objective, the child's performance of objectives C and D can give some assurance that the numerals the child works with are tied to a basic concept of number and set size.

Objective 5:E requires the comparison of rows of objects deliberately arranged so that length and number are uncorrelated. For example, in successive test items for this objective, the longer row might have fewer objects, the longer row more objects, two rows of equal length might have different numbers of objects, and two rows of unequal length might have an equal number of objects. Successful performance of this task requires that the child attend to number as a dimension independent of length. Thus, the objective constitutes a somewhat unorthodox test of conservation of number (Piaget, 1965).

A more usual test of conservation is to present two sets of objects, paired in one-to-one correspondence, and obtain agreement from the child that the sets are equal in number. One of the rows is then contracted, expanded, or otherwise rearranged, with the child watching, and the child is then asked whether the sets still have the same number. Non-conserving children do not recognize that equivalence of number is maintained despite spatial transformation.

This test, along with most tests developed for laboratory study of conservation behavior, can be easily invalidated by teaching.[3] With enough rehearsal, the child will undoubtedly learn to state, "They still have the same number", after rearrangement; but there is every chance that he will merely be saying what he knows the teacher wants to hear. Although a minor problem in the laboratory, where rehearsal is usually deliberately avoided, this would be a serious weakness were the laboratory task to be used directly in an educational curriculum, particularly a "mastery" curriculum in which teachers are encouraged directly to "teach for" each specified objective.

The task specified in Objective 5:E is not subject to this problem. A large number of different test and practice items for the objective can be prepared, and each new item presented will require that the child figure out for himself which row has more objects. If he believes that longer (or denser) rows always have more, the teacher will surely discover it. This particular test of number conservation was chosen because, in a pilot experiment, it showed a strong correlation ($r = 0.77$) with the standard test of number conservation described above. More formal experiments to validate this finding are now underway.

Figure 10 shows the analysis of Objective 5:E. There are two alternative methods by which the child can solve the problem posed by this task. In the "counting method" (box IIa), he counts each set separately and then compares the stated numbers. This is equivalent to Objective 5:A, to which the reader is referred (box IVa). The "one-to-one correspondence method" (box IIb) requires that the child visually "pair" the objects in the two rows, and then determine whether there are "extra" items in either set. With the exception of the components of visually pairing the objects (box IIIb) and remembering which have been paired (box IVb), this process is the equivalent of Objectives G and H in Units 1 and 2, which are therefore referenced in box Va. However, it should be recognized that the process of visual pairing, with its concomitant memory demand (box IVc), substantially increases the difficulty of the task and may be one of the reasons that young children tend strongly to respond to the physical shape of the array in conservation tests.

In Objective 5:F, the child must compare several sets, selecting the one with the most (or fewest) objects. The task analysis for this objective showed a process of successive comparison. Two sets are compared and the largest selected; then the selected set is compared with the third set, and the largest of these two se-

[3]For a critique of experimental tests of conservation, see Rothenberg (1969).

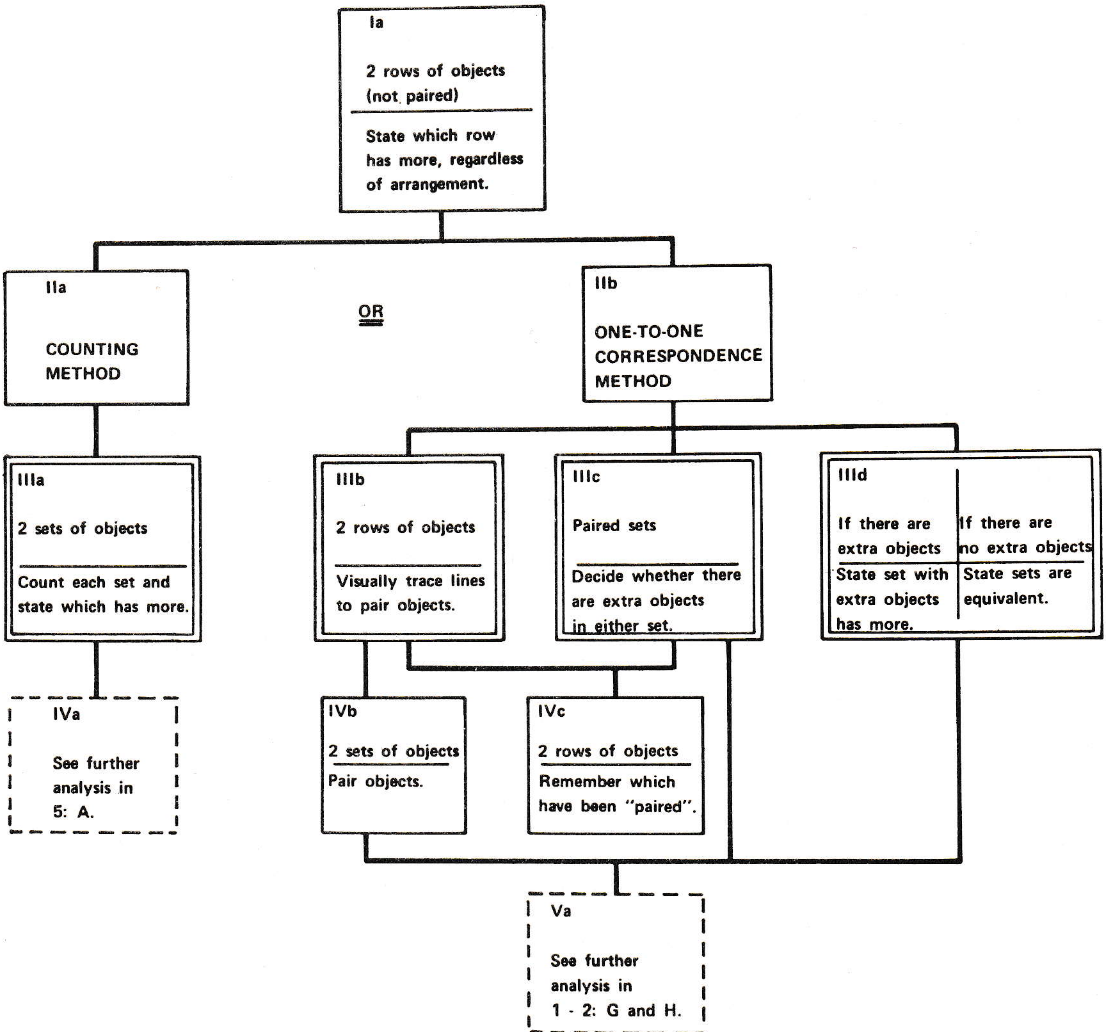

Fig. 10. Analysis of Objective 5:E.

lected. The process is analogous to the one already described as a component of ordering numerals.

Seriation: Unit 6

A child's ability to seriate sets according to numerosity (Objective 6:C) demonstrates his comprehension of the ordered relationship among sets of different numerosity, and thus is yet another indicator of the child's possession of an operational number concept. Seriation by size (Objective 6:B) and by numerosity jointly provide the basis for eventually establishing correspondence between ordinal and cardinal number. This ability is treated as an important aspect of the number concept by Piaget (1965), although in America it has been almost completely overshadowed by conservation as a topic of interest to developmental psychologists.

There are at least two different methods for performing the seriation task. One method is to select the largest (or smallest) of the array, then the largest (or smallest) of those remaining, and continue until all items have been selected and placed. This is the method of "operational seriation" described by Inhelder and Piaget (1964). Figure 11 shows the analysis of this method for seriating objects. Boxes IIIb,

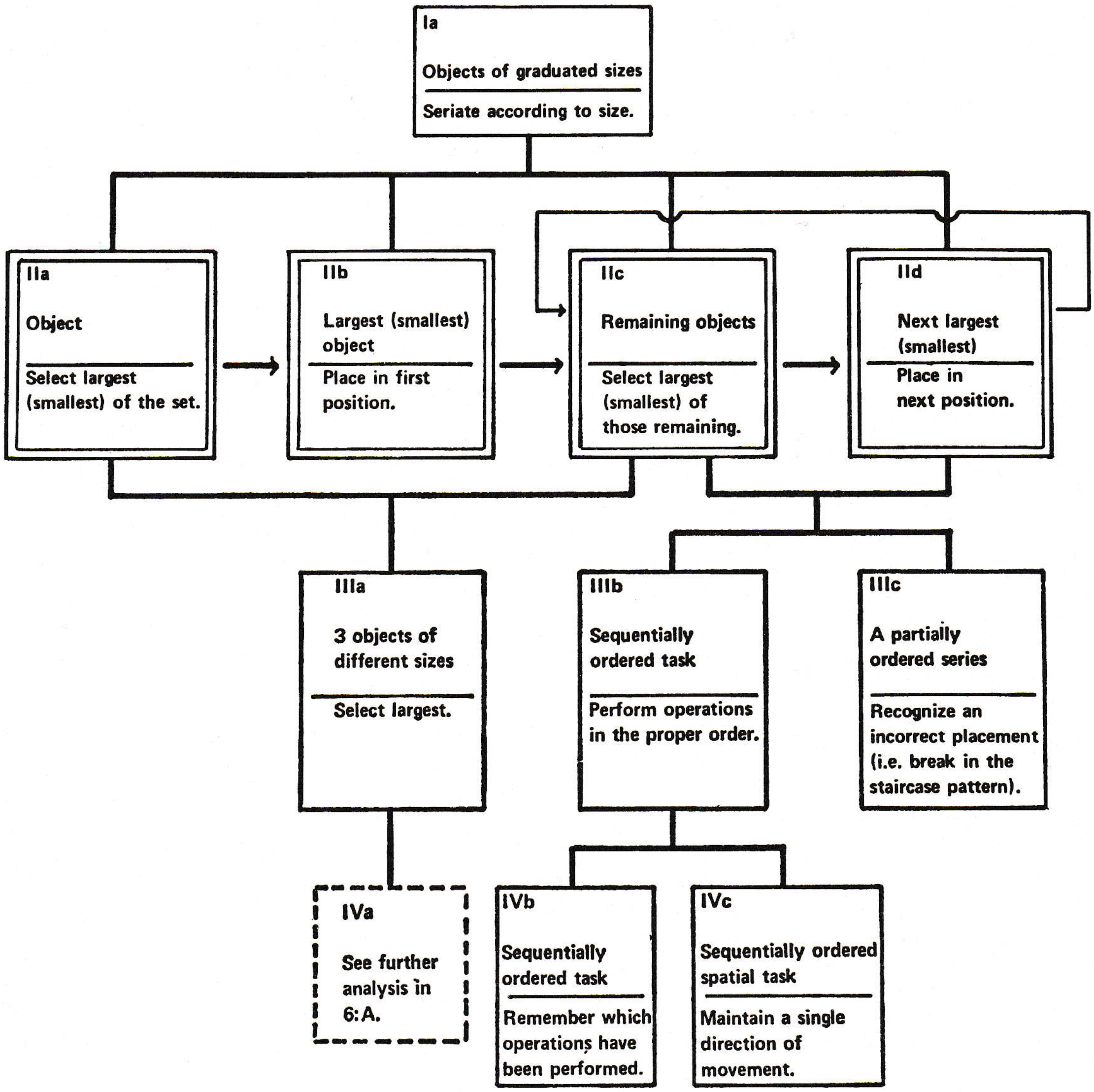

Fig. 11. Analysis of Objective 6:B (Alternate 1).

IVb, and IVc describe a set of prerequisites concerning the performance of sequential operations. These are common to size seriation and set seriation. An additional hypothesized prerequisite for size seriation is the ability simply to recognize a misordering (box IIIc). According to our informal observations during attempts to teach seriation directly, many children who cannot seriate also lack this ability.

The sharpest difference between size and set seriation seems to lie in the process of selecting the largest in the array. Selection of the largest size object can be accomplished by direct perceptual inspection, which permits comparison of several objects virtually simultaneously. Selection of the more numerous set, however, requires successive comparisons of pairs of sets. Successive rather than simultaneous comparison is also required for size seriation when the task is performed tactually rather than visually, or when the differences between adjacent sizes are so slight as to require direct measurement. Tactual seriation is more difficult than visual seriation (Inhelder and Piaget, 1964). By analogy, it is reasonable to expect set seriation to be more difficult than visual size seriation. In addition, selection of the more numerous set requires operations of counting and of remembering numbers while counting, neither of which is required for size seriation. Thus, a reasonable prediction is that learning size seriation first will facilitate, but not directly produce, learning to seriate sets.

Figure 12 shows an analysis of a second method of seriation. Using this method, the child orders two objects or sets, then places a third item with respect to the first two. He continues placing new items until all items have been ordered. A primitive form of transitivity operates in this solution, in that the child need not directly compare each new set with all sets already ordered. As shown in box IIe of the figure, he stops as soon as he finds a set smaller than the new set he is trying to place, assuming that all subsequent sets will also be smaller. Of course, at an early stage in learning, the child might indeed make many logically unnecessary direct comparisons. However, in skilled performance of the seriation task, the extra comparisons should drop out.

As in the first method, the size and set seriation tasks share prerequisites concerned with spatial organization and maintenance of sequence. However, set seriation requires, in addition, counting and memory functions, and thus should be the more difficult skill to acquire.

The two methods of seriation described here for ordering according to size and numerosity are directly analogous to the two methods identified earlier for ordering numerals (Objective 3-4:F). The same methods could be applied to problems of ordering weights, color intensities, or other dimensions. Thus, the logical operations of seriation are not restricted to size or numerosity, and considerable positive transfer from one seriation task to another can be expected. There is some reason to believe that the second method, which requires successive comparisons, is the more generalizable, since, logically, it would not need to be modified to apply to problems (such as tactual seriation or weight seriation) in which simultaneous perceptual comparisons of several objects were impossible. This hypothesis, however, is in need of a direct empirical test.

Addition and Subtraction: Units 7 and 8

Unit 7 introduces the concepts of union and partition of sets, in the form of addition and subtraction. These concepts are included in the introductory part of the curriculum in order to round out and stabilize the child's concept of set and number and to prepare him for a more abstract stage of mathematical understanding. Children who learn to count reliably under various conditions, as in Units 1 and 2, and who learn the relation of counting to other components of the number system, as in Units 5 and 6, often seem to move naturally into addition and subtraction. For these children, an expanded definition of "four" can include the fact that it can be made of two "two's", or of a "three" and a "one", and, later, that two "fours" can be combined to make an "eight". The aim of this unit is to develop these basic concepts rather than to have the child memorize the addition and subtraction combinations.

To implement this goal, Unit 7 contains objectives that specify two different methods of adding and subtracting. In Objectives A and B, the child learns to use "counters" (these could be tally marks as well as counting blocks, chips, or other objects) to establish sets and then unite (A) or partition (B) them. In Objectives C and D, number is translated into length as the child uses a number line in his calculations. The analyses of these skills suggest that using a number line is a more complex task than using counters. As shown in Figure 13, the number line requires basic spatial organization skills (box IIIc) in addition to appropriate use of the "zero" position (box IIIa), and the reading of numerals. None of these behaviors is directly called for in adding or subtracting with counters. It is likely, therefore, that Objectives A and B will be learned more easily than C and D.

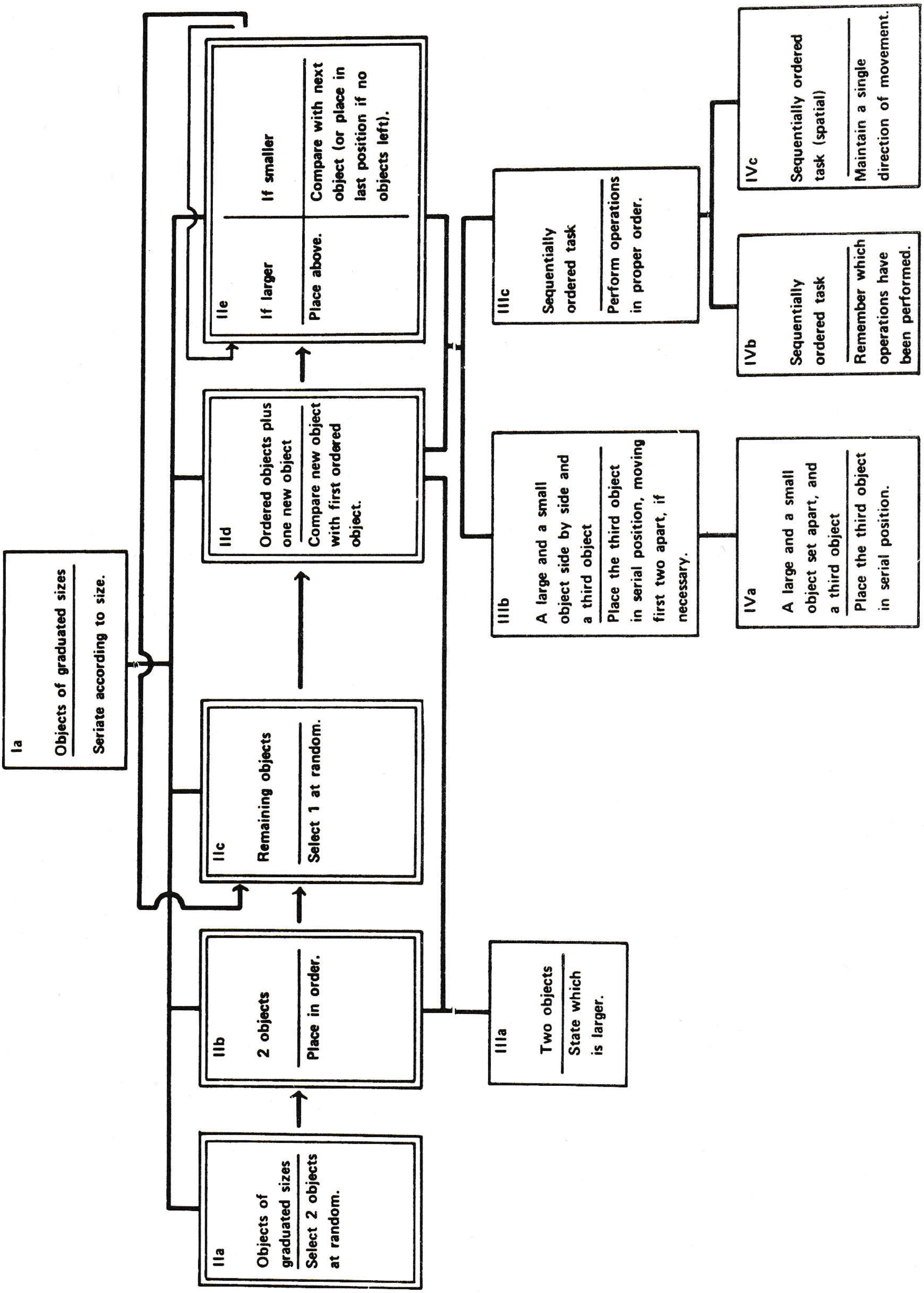

Fig. 12. Analysis of Objective 6:B (Alternate 2).

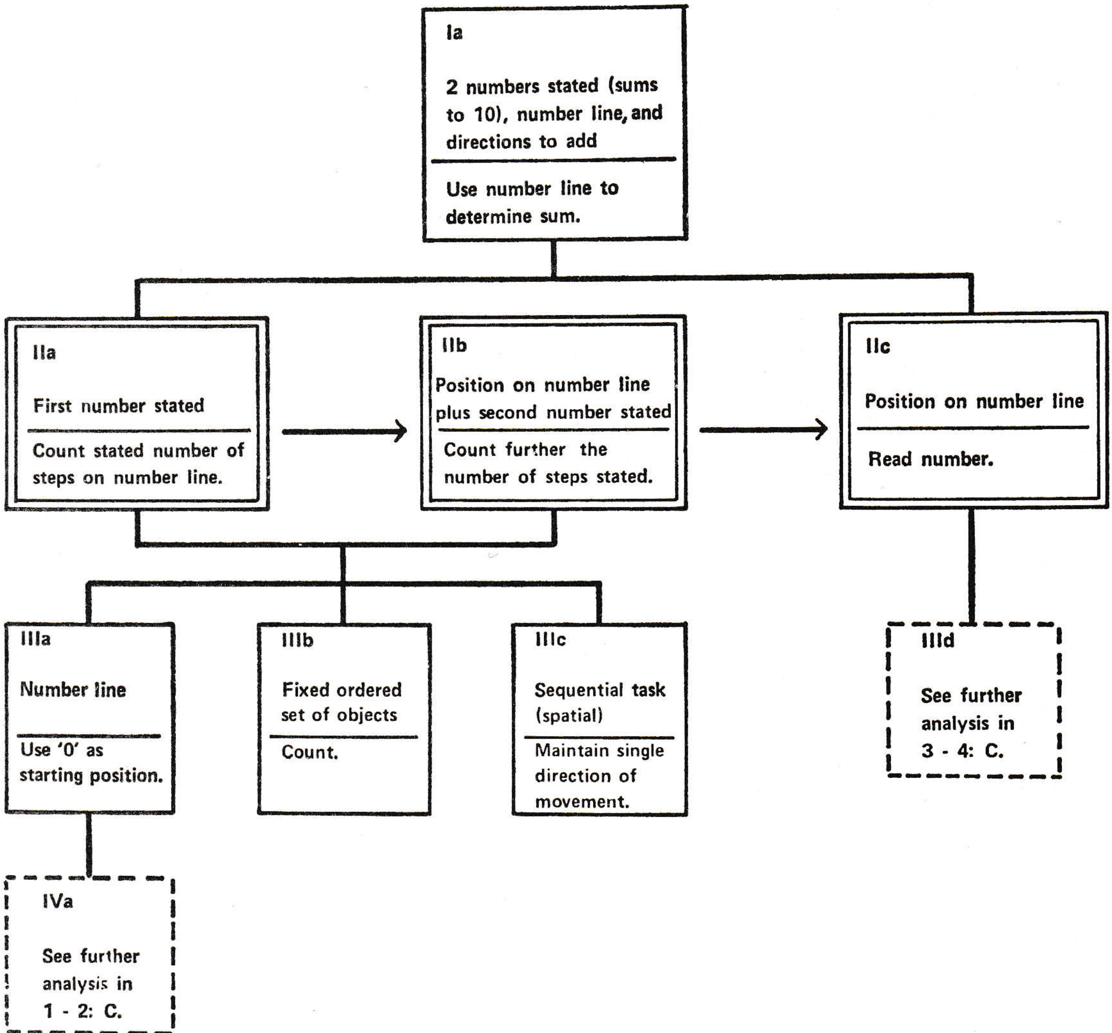

Fig. 13. Analysis of Objective 7:C.

However, since the two processes seem quite independent, in the sense of having few common prerequisites, they have been treated as separate branches within the unit. Should later studies of hierarchical relationships among these objectives suggest that learning A and B first would strongly facilitate learning C and D, these objectives would be combined into a single linear sequence.

Only after the basic concepts of addition and subtraction are established does the curriculum introduce word problems and written formats (Objectives E, F, and G) as specific objectives. Objectives F and G require a straightforward reading of symbols, and have not been separately analyzed. Solving "word problems" (Objective E), however, is frequently quite difficult even for children who can solve symbolically presented addition and subtraction problems. These children have difficulty in translating the verbal statement into a familiar and solvable addition or subtraction problem. Experimental analyses of tasks of this kind are underway in our laboratory (*e.g.*, Rosenthal, *unpublished*) preparatory to experiments in teaching children to solve verbally presented mathematics problems.

For many children, written equations or word problems may be the best way of giving instruction in Objectives A through D. These children will pass Objectives E, F, or G simultaneously with A to D. However, the separation of concept from symbolization in the formal curriculum permits children who need to work on one problem at a time to do so, and to experience measurable success at an early stage.

The expansion of equation formats in Unit 8 is not simply a matter of algebraic virtuosity. Rather, each step in the sequence is designed to direct the child's attention to some basic mathematical concept. It is assumed that counters or a number line will continue to be used, both as an aid to calculation and as a means of highlighting the number concept underlying the algebraic processes. Objectives A and B, for example, are intended to show the child that there are many ways of composing a given number. They also provide occasion for demonstrating the fact that $x + y$ is always equivalent to $y + x$, the rule of "commutativity", although this rule need not be formally learned at this stage. Objective D (with C as a transition) requires the child to complete an equation with one addend plus the sum given. This is very difficult for young children, and requires considerable flexibility in the manipulation of addition concepts. One way of performing the task is to treat it as a subtraction problem. To highlight the addition-subtraction complementarity, Objective E has been placed at the same level as D, suggesting that the two objectives be taught simultaneously. E requires the child to construct subtraction equations that are complementary to a given addition equation. This can be done by either a "counter" or a "number line" method for demonstrating the relationship. In Objective F, the child is freed from pre-set problems; he now composes equations in all the formats he has experienced. With this objective, the child can be assumed to have developed a self-monitored control over number operations.

IMPLEMENTATION AND STUDY OF THE CURRICULUM

The curriculum presented here provides an organized set of learning objectives around which instructional programs of many types can be organized. The particular form of instruction —group *versus* individual; "programmed" *versus* "discovery", *etc.*—is not specified. This omission is deliberate. The important question in a mastery curriculum is not *how* an objective is taught, but whether it is learned by each child. On this view, the school's job is to assure that all children do learn, regardless of time needed or specific teaching method. In this work, a carefully sequenced curriculum is one of the essential tools.

In practice, implementation of a mastery curriculum implies that children will be permitted to proceed through the curriculum at varied rates and in various styles, skipping formal instruction altogether in skills or concepts they are able to master in other ways. This demand for individualization, in turn, requires that there be some method of assessing mastery of the various objectives in the curriculum. If children are to work only on objectives in which they need instruction and for which they are "ready", in the sense of having mastered major prerequisites, then teachers need to feel considerable assurance that mastery has in fact occurred.

In our classrooms, the need for assessment is met through frequent testing and systematic record keeping. A brief test for each objective in the curriculum has been written.[4] These tests directly sample the behavior described in the objective. If the objective is counting objects, for example, the child is given sets of objects to count. If the objective involves seriating rods, he is given rods to place in order. The test informs the teacher of the presence or absence of the behavior in question. Thus, the test items are a direct reflection of the curriculum objectives and define very explicitly what the child is expected to learn.

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[4]The tests are available for research use from Dr. Margaret C. Wang, Learning Research and Development Center, University of Pittsburgh, Pittsburgh, Pennsylvania 15213. A charge of $5.00 covers costs of printing and handling.

Assessment

After a child is socially comfortable in the classroom and routines are well established, the teacher or aide takes him aside and begins the testing program. The first task is to find his "entering level". This is normally done by administering a special "placement test", composed of a sampling of items from the units. Children can be rated as passing or failing each unit on the basis of this test. For units failed, tests on the individual objectives may then be administered to determine exactly which objectives the child needs to work on. The placement testing procedure is an efficient one in terms of testing time, especially for groups in which the entering levels of individual children are expected to spread over a wide range. An alternate procedure is to administer the unit tests themselves, beginning with Unit 1 and moving up through subsequent units until the child stops passing tests. This is the point in the curriculum in which instruction should begin.

When a child does not pass a test, indicating that he needs work on a given objective, he is given one or several "prescriptions", or assignments, of activities relevant to learning that objective. Prescriptions in the mathematics curriculum are extremely varied. For independent work by children, they range from interactive games for two or more children to formal written worksheets. Small group and individual "tutorials" with the teacher are also prescribed when needed. Conceptual mathematics teaching materials such as those developed by Montessori (1965), Dienes (1967), and Cuisenaire (see Gattegno, 1963) are used, along with material from virtually every major educational supply house in America. Audio-visual devices such as the Language Master and Audio Flashcard machines are used, and other devices are being investigated. Each teacher also continues to develop many materials on her own to meet specific needs.

When a child has completed prescribed work on an objective, he is retested, and, if necessary, further instruction is provided until mastery is demonstrated. A child may work on several different objectives during a given instructional period, working up independent branches of the hierarchy. As the child moves through the curriculum, a pre-test on each new objective or unit assures that he will be allowed to skip over objectives he has been able to learn on his own.[5] The testing program serves the teacher as a constant check on her success in teaching. The test outcomes also provide our research and development staff with a data base for continued study and evaluation of the curriculum. In addition, they can serve as a dependent variable in classroom research of other kinds.

Some examples of the kind of data collected, and the way in which it is interpreted, will help to clarify the research role of the hierarchically sequenced testing program. The graphs in Figures 14 to 19 display varying patterns of progress through the curriculum for eight kindergarten children over the course of a school year. Objectives mastered during each month are marked with an *X*, yielding a relatively detailed picture of the pattern of mastery shown by each individual. The inset graphs are summaries that show only the number of new objectives mastered. They are thus more traditional cumulative records, and permit some comparison of subjects.

As can be seen from inspecting the inset graphs, there is wide variation among children in the number of objectives mastered by the end of the year. There is also considerable variation in the rate patterns. Marie (Figure 14), for example, shows a relatively steady, high rate; and Roy (Figure 16), a steady but slower rate; Robert (Figure 15), on the other hand, placed

[5]The effectiveness of the general procedure can be estimated from data on the use of the program in our developmental classrooms in an inner-city school. In 1970, kindergarten children using the program had a median grade equivalent of 1.5 in the Wide Range Achievement Test. First graders, also in the program, showed a median of 2.4. There was no formal control group, but the performance of second-grade children in the same school who had never used the program provides a rough comparison. These children's median grade equivalent was 2.3, slightly lower than that of the first graders.

informal "control" group

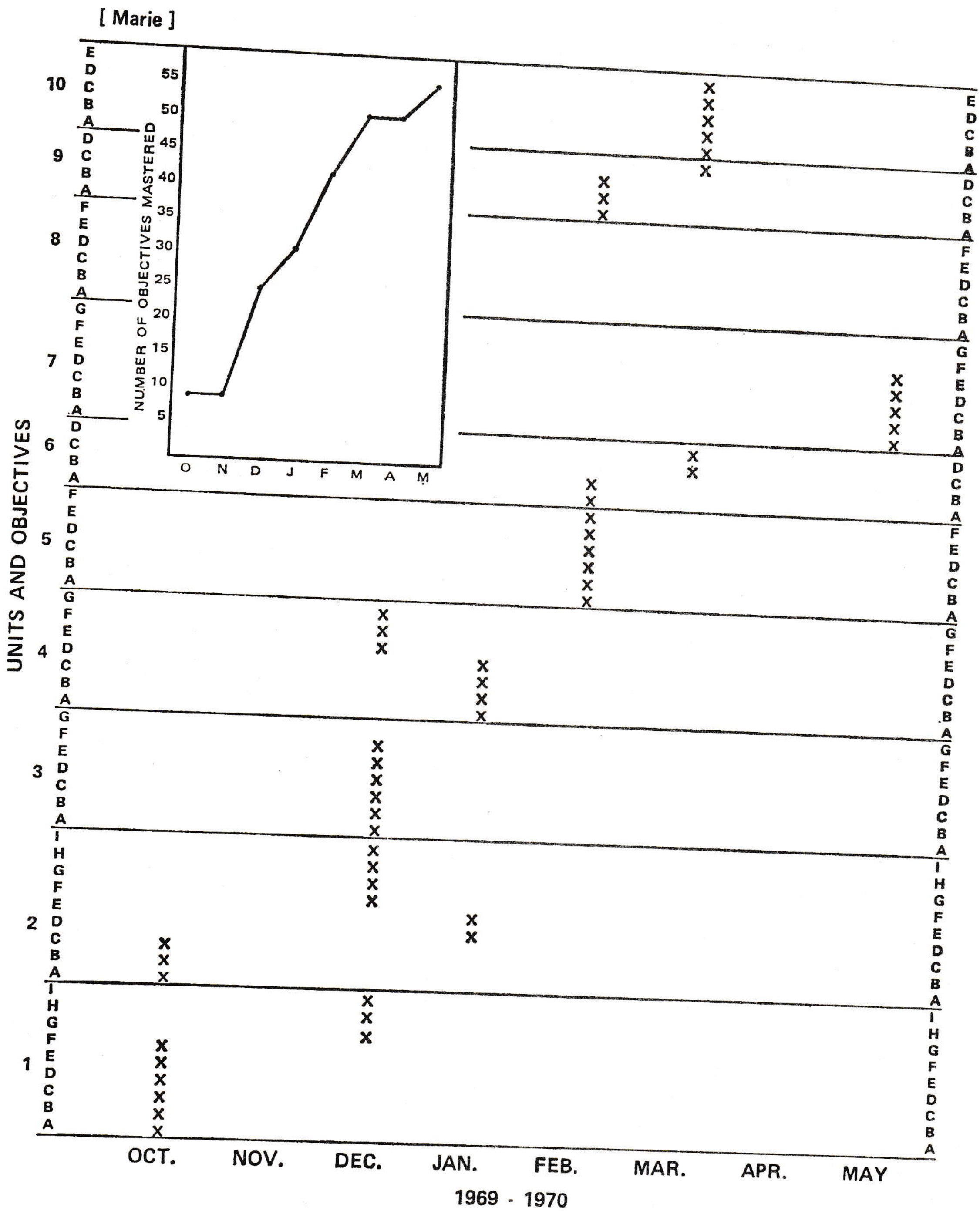

Fig. 14. Individual mastery pattern (Marie).

high in the curriculum early in the year, but thereafter proceeded at a rather modest rate. His performance suggests that he had already been exposed to most of the lower-level concepts before entering kindergarten. Kenneth (Figure 17), too, made very rapid progress during the first part of the year, but his rate of mastery thereafter was not different from that

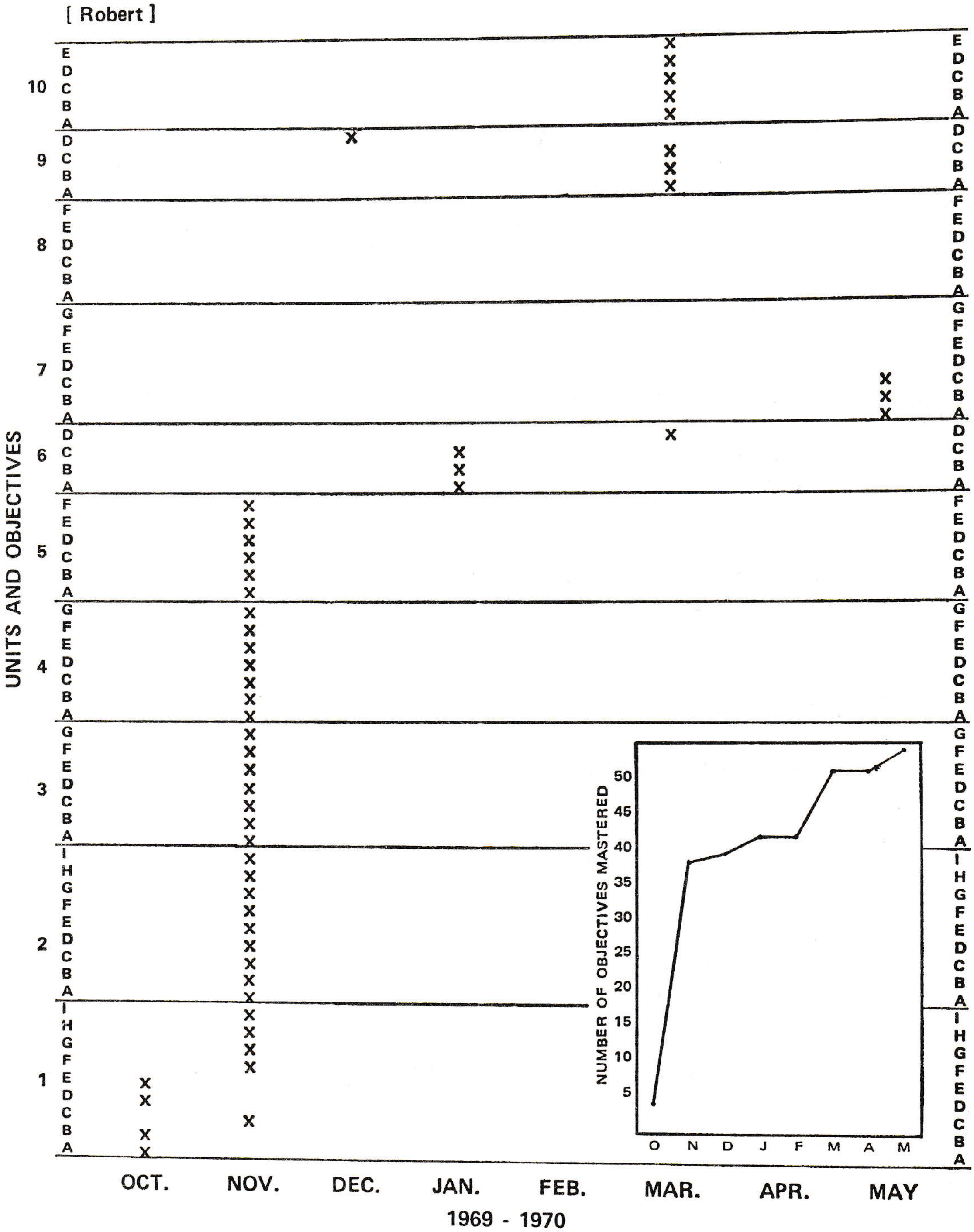

Fig. 15. Individual mastery pattern (Robert).

of George (Figure 19), the slowest student in the group shown.

Examination of the main, detailed graphs gives information concerning the order in which objectives were learned and the parts of the curriculum that seemed most difficult to learn.

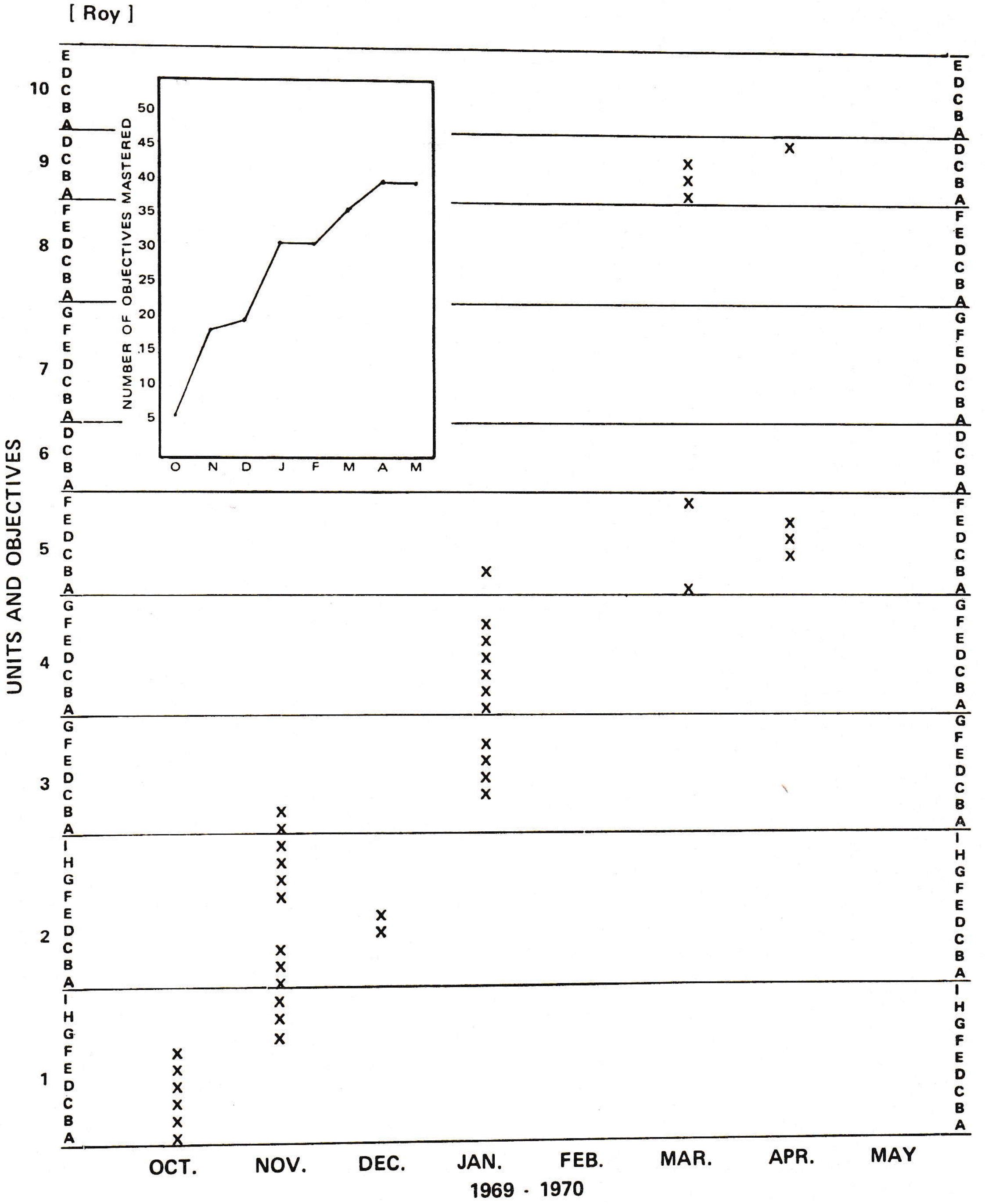

Fig. 16. Individual mastery pattern (Roy).

This information is mainly important for the further study and modification of the curriculum, but it also suggests some limitations on interpretation of the rate data discussed above. The graphs reveal that the order of progression was rarely "linear". That is, children often began new units before all preceding ones were completed, and some units were typically

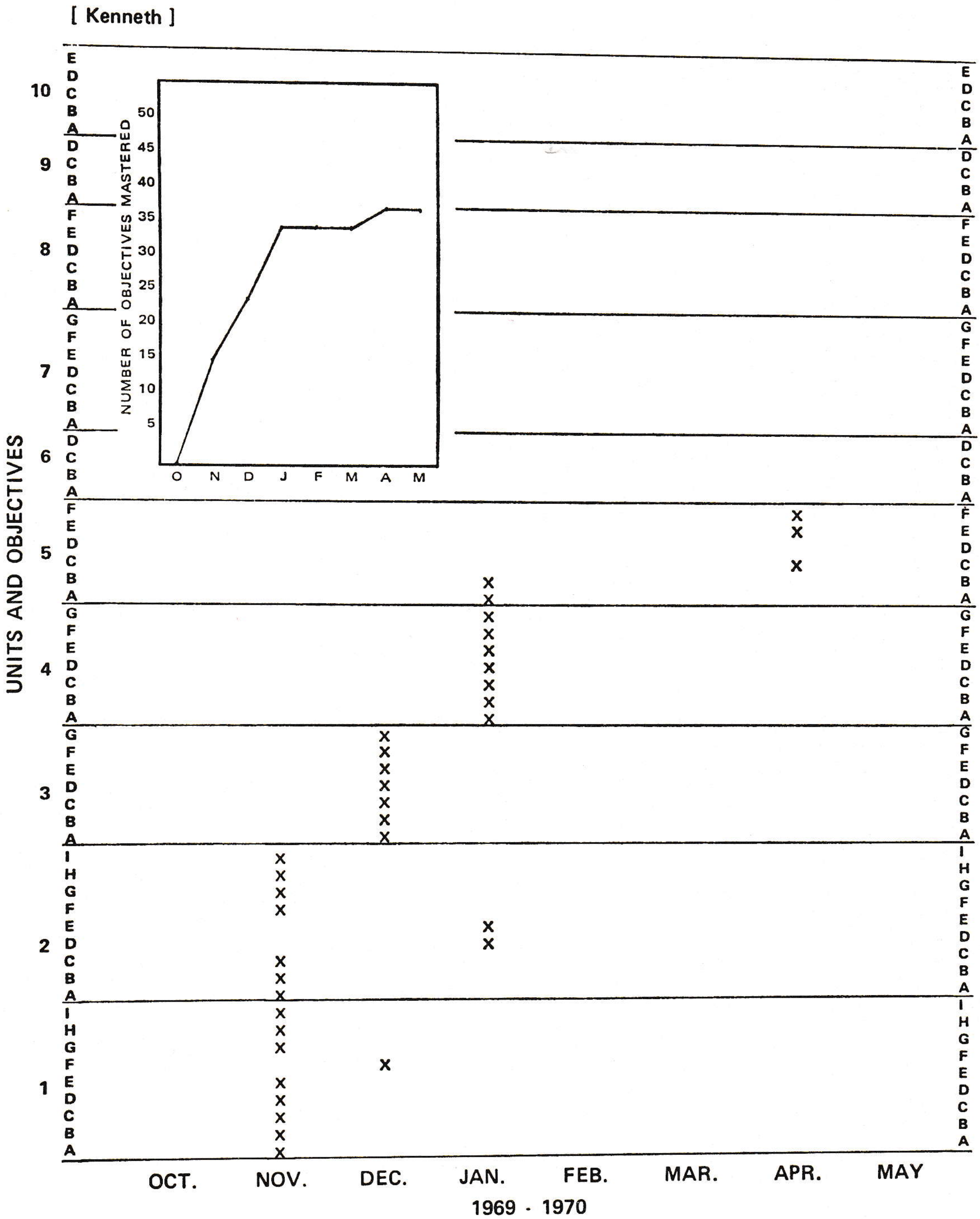

Fig. 17. Individual mastery pattern (Kenneth).

learned out of "numerical order" (*e.g.,* units 9 and 10 before 8). With only a few exceptions, however, there were no violations of the orders permitted by either the within-unit or between-unit hierarchies. While the infrequency of such violations supports the validity of the hierarchical sequences in a general way, it cannot be counted as very firm evidence in favor of the

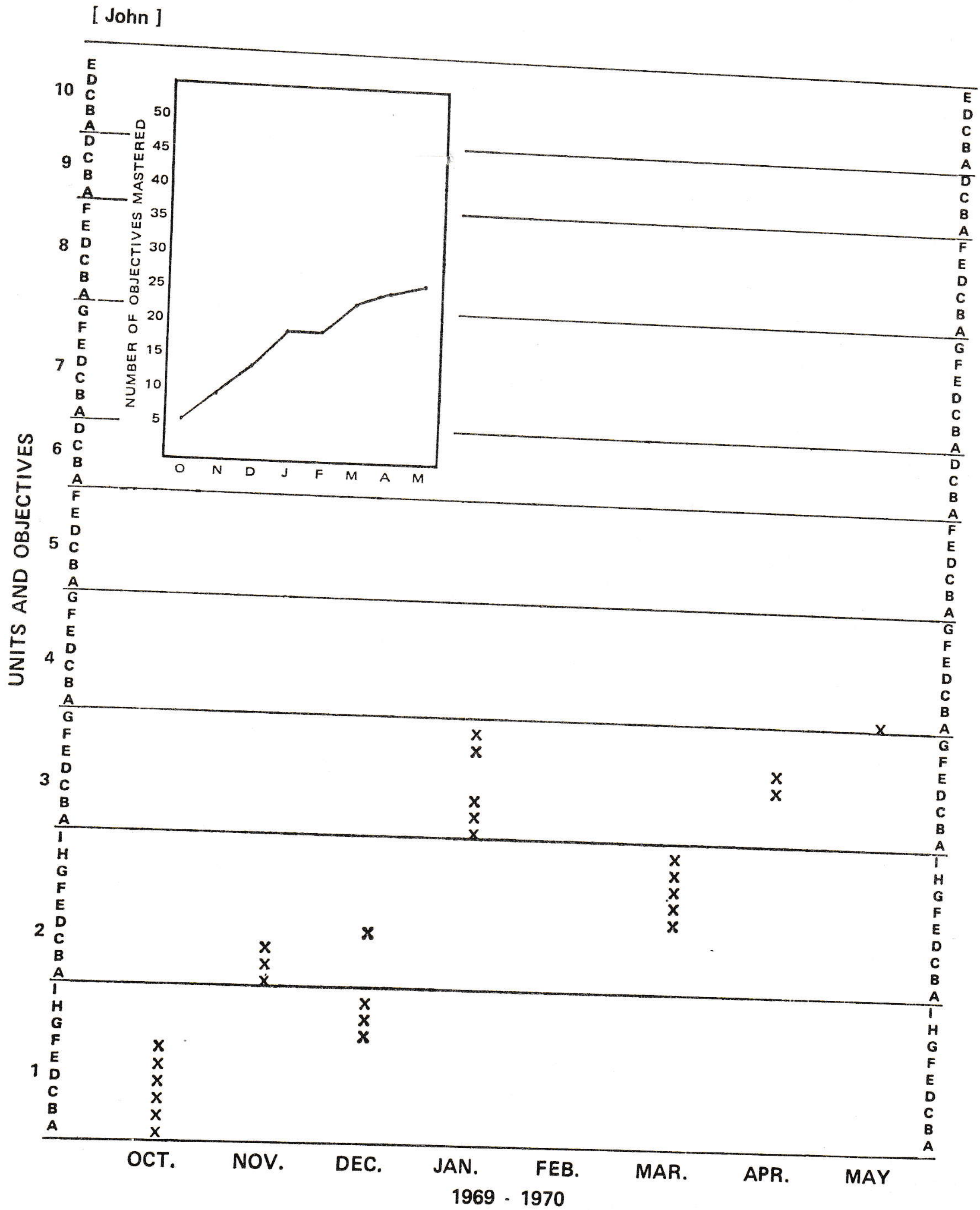

Fig. 18. Individual mastery pattern (John).

details of the hierarchy. This is because teachers typically prescribed the units and objectives in the orders specified as permissible, and children therefore learned them in those orders. Validation of the hierarchical sequences requires other forms of experimental evidence (see Resnick and Wang, 1969; Resnick, *in press*), and such studies are in fact being conducted

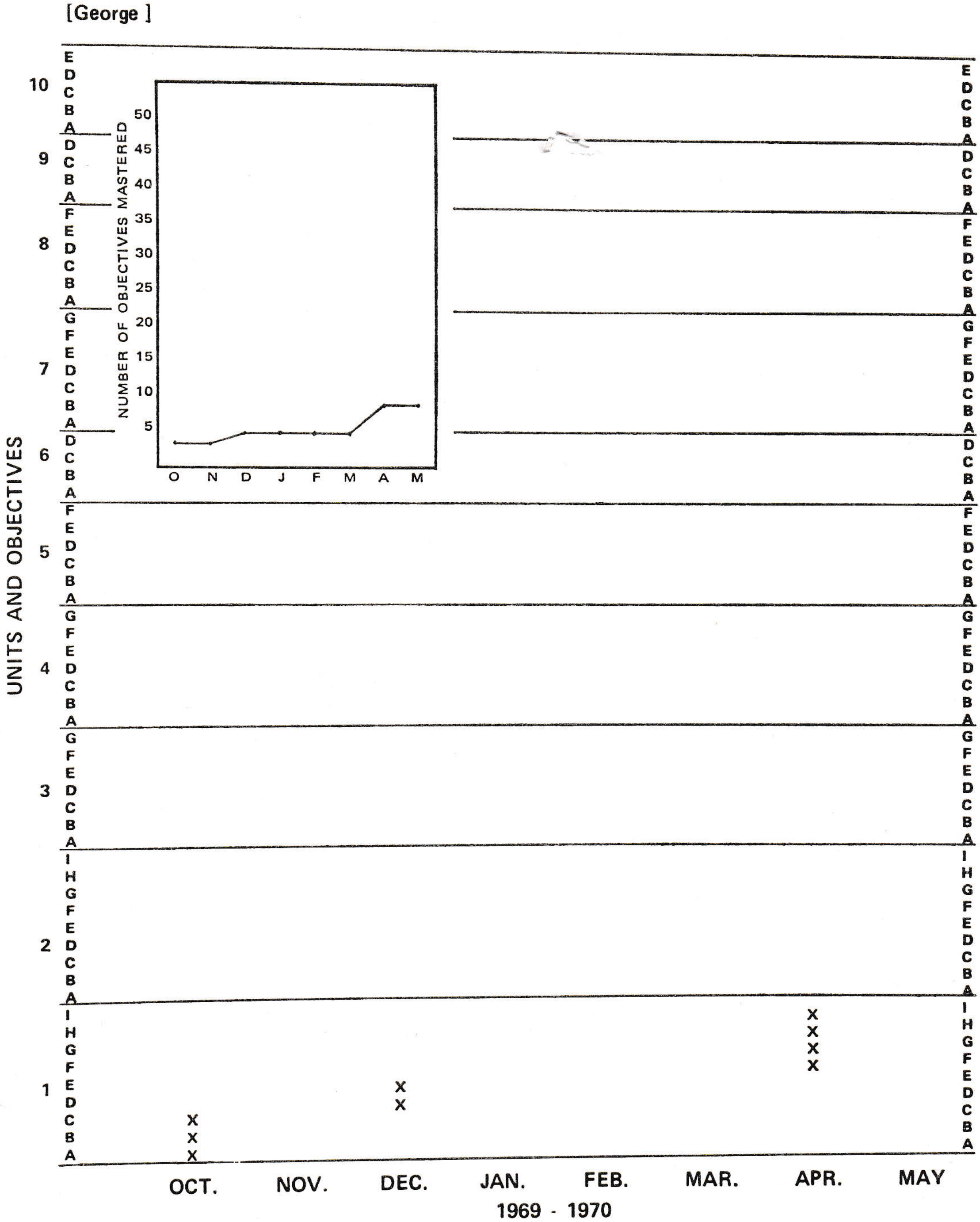

Fig. 19. Individual mastery pattern (George).

with respect to the present curriculum (*e.g.*, Wang *et al.*, 1971; Wang, 1973).

The present data do, however, permit identification of objectives or units that are typically learned slowly and with difficulty. For example, mastery of the early objectives dealing with one-to-one correspondence of sets (G-H-I in Units 1 and 2) is often delayed longer than mastery of counting (B-F in Units 1 and 2). Figures 14 and 18 illustrate this lag. In one case (John, Figure 18), the child was well along in learning numerals (Unit 3) before the one-to-one correspondence objectives of Unit 2 were mastered. For the children who progressed beyond Unit 5, there was in most cases a slowing down of rate of acquisition as the child entered Units 6 and 7. The relatively greater difficulty of these units is reflected in a flattening out of the inset rate curve at the corresponding point in time.

Other general features of the curriculum that can be noted in the present data are the extreme difficulty of Unit 8, such that none of the children had mastered its objectives by the end of the kindergarten year, and the relative speed with which Units 9 and 10 were learned by those children attempting them. Findings of this kind suggest the possibility of reordering these units. Equally as important, they dictate careful re-examination of the objectives and related teaching techniques of Unit 8. This is precisely the kind of work that we are now engaged in, and a much more finely differentiated set of objectives on equations, together with fuller attention to certain prerequisites, is being studied as a means of meeting the difficulties encountered in Unit 8.

Data of this kind, which can be collected and plotted continuously through a school year, can provide a dependent variable for classroom research of various kinds. In the absence of valid measures of learning outcomes that can be administered sequentially throughout an instructional program, much classroom research has focused on measures of task attention or "non-disruptive" behavior, rather than on learning itself. The testing program outlined here provides an alternative. The validity of the sequentially administered mastery test results as measures of the mathematical competencies typically required by schools can be estimated from the correlations of number of objectives
mastered at the end of a school year with end-of-year scores on arithmetic achievement tests. For two successive years in which the curriculum described here was in use, correlations for kindergarten children with the Wide Range Achievement Test (Jastak and Jastak, 1965) arithmetic sub-test were 0.61 and 0.76.

When used in classroom research, the data from the hierarchy tests must be interpreted with some caution, due to the unequal difficulty of the units. For example, a slowing of rate of mastery at the point of entering Unit 6 cannot automatically be interpreted as due to a change in some non-curricular independent variable under study. More generally, even if all other features of the environment are held constant, mastery curves should not be expected to be perfectly regular, since the interval or "step size" between objectives is not exactly the same at all points in the hierarchy. While a reordering or regrouping of objectives, or the addition of some transitional ones at the more difficult points in the curriculum, would correct this problem to some extent, none of these measures can be expected to eliminate it. The problem of unequal units is, rather, inherent in any situation in which a new behavioral repertoire is being shaped and studied, as opposed to one in which repeated occurrences of the same set of behaviors are observed. A testing program, such as the present one, based on careful analysis of the tasks to be learned, can contribute substantially to a reduction of the measurement problem in educational settings. However, careful examination of individual cases with respect to what is known about general characteristics of the curriculum will continue to be necessary in interpretation of the data.

As these data and the accompanying discussion perhaps make evident, implementation of a behaviorally designed curriculum in a school does not mark the conclusion of a research or curriculum development program. Rather it creates a "laboratory" in which empirical study of the curriculum and the effects of other classroom variables can proceed while, at the same time, children's immediate needs are met. Thus, the curriculum outlined here should be regarded as still under study and development. By

correlations of WRAT scores
+ # objectives met.

reporting it at this intermediate stage, we hope to provide both a practical guide for educators seeking to develop a systematic early learning program and a basis for continuing exchange among researchers interested in questions of early mathematics learning and teaching.

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Reviewer's Comments on next page

REVIEWER'S COMMENTS

The manuscript by Resnick, Wang, and Kaplan is a very interesting example of how a "cognitive" skill can be analyzed in behavioral terms. The authors have laid out in a very reasonable fashion the steps, components, and objectives of an introductory mathematics curriculum. It is a remarkable example of fine-grain analysis of an abstract and important concept (numerosity), in very worthwhile and comprehensive levels of its operation (addition, subtraction, equations, *etc.*). The analysis of a "concept" such as that (or those) into objective behaviors is a worthwhile example for quite a lot of our audience. The probably clever sequencing within this analysis also adds to its educational value, as an example of good programming *per se.* The relevance to classroom procedure is very good for a large part of our audience.

Some discussion is included describing how the data that are accumulated from the continuing testing program could be used as a dependent variable for classroom research dealing with both curricular and non-curricular variables. One of the most difficult problems of classroom research today is the lack of a standard dependent variable. Standardized tests that sample skills only once a year are clearly insensitive and inadequate. The behavioral analysis provided by Resnick, Wang, and Kaplan, together with Wang's materials, could very well be a large step toward a sensitive and comprehensive continuous measure of skill acquisition in mathematical behavior.

On the negative side, it presents no experimental data, and requires *many* pages. It may invite a flood of similar articles, almost certainly of lesser quality, which *JABA* would be disposed not to accept. However, this one Invited Article makes an example of considerable value, which may prompt experimental analyses of academic curricula.